The Mandelbrot Set and Julia Sets

"Run Away to Infinity" Criterion

Here we show that if some z_n is farther than 2 from the origin, then successive iterates will grow without bound. That is, they will run away to infinity.

For a complex number $z_n = x_n + i*y_n$, the absolute value is

$$|z_n| = \text{sqrt}(x_n^2 + y_n^2),$$

the distance from z_n to the origin.

Recalling the sequence $z_0, z_1, \dots$ is defined by $z_{n+1} = z_n^2 + c$, we show if some z_n satisfies $|z_n| > \max(2, |c|)$, then the sequence $z_n, z_{n+1}, \dots$ runs away to infinity.

So suppose $|z_n| > \max(2, |c|)$.

Because $|z_n| > 2$, we can write

$$|z_n| = 2 + e,$$

for some $e > 0$.

Now

$$|z_n^2| = |z_n^2 + c - c| <= |z_n^2 + c| + |c|$$

So

$$|z_n^2 + c| >= |z_n^2| - |c| = |z_n|^2 - |c|$$

$$> |z_n|^2 - |z_n| \text{ (because } |z_n| > |c|)$$

$$= (|z_n| - 1)*|z_n| = (1 + e)*|z_n|$$

That is, $|z_{n+1}| > (1 + e)*|z_n|$. Iterating, $|z_{n+k}| > (1 + e)^k*|z_n|$.

To complete the proof that $|z_n| > 2$ implies the sequence runs away to infinity, observe that if $|c| > 2$, then

$$z_0 = 0$$

$$z_1 = c$$

$$\text{and } z_2 = c^2 + c = c*(c + 1)$$

$$\text{so } |z_2| = |c|*|c + 1| > |c| \text{ (noting } |c + 1| > 1 \text{ because } |c| > 2).$$

Return to JuliaSets.