CS580:
Monte Carol Integration

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Course URL:
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KAIST

Class Objectives

- Sampling approach for solving the rendering equation
  - Monte Carlo integration
  - Estimator and its variance
  - Sampling according to the pdf

Two Forms of the Rendering Equation

- **Hemisphere integration**

$$L(x \rightarrow \Theta) = L_e(x \rightarrow \Theta) + \int_{\Omega_x} f_r(\Psi \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi) \cdot \cos\theta_x \cdot d\omega_\Psi$$

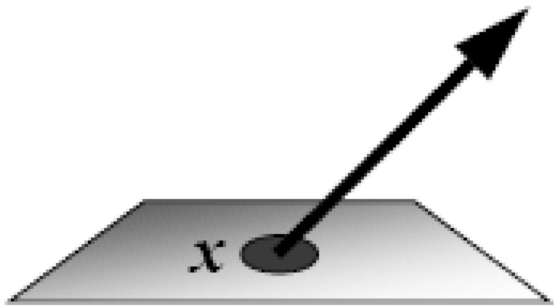
- **Area integration**

$$L(x \rightarrow \Theta) = L_e(x \rightarrow \Theta) + \int_A f_r(\Psi \leftrightarrow \Theta) \cdot L(y \rightarrow -\Psi) \cdot \frac{\cos\theta_x \cdot \cos\theta_y}{r_{xy}^2} \cdot V(x,y) \cdot dA_y$$

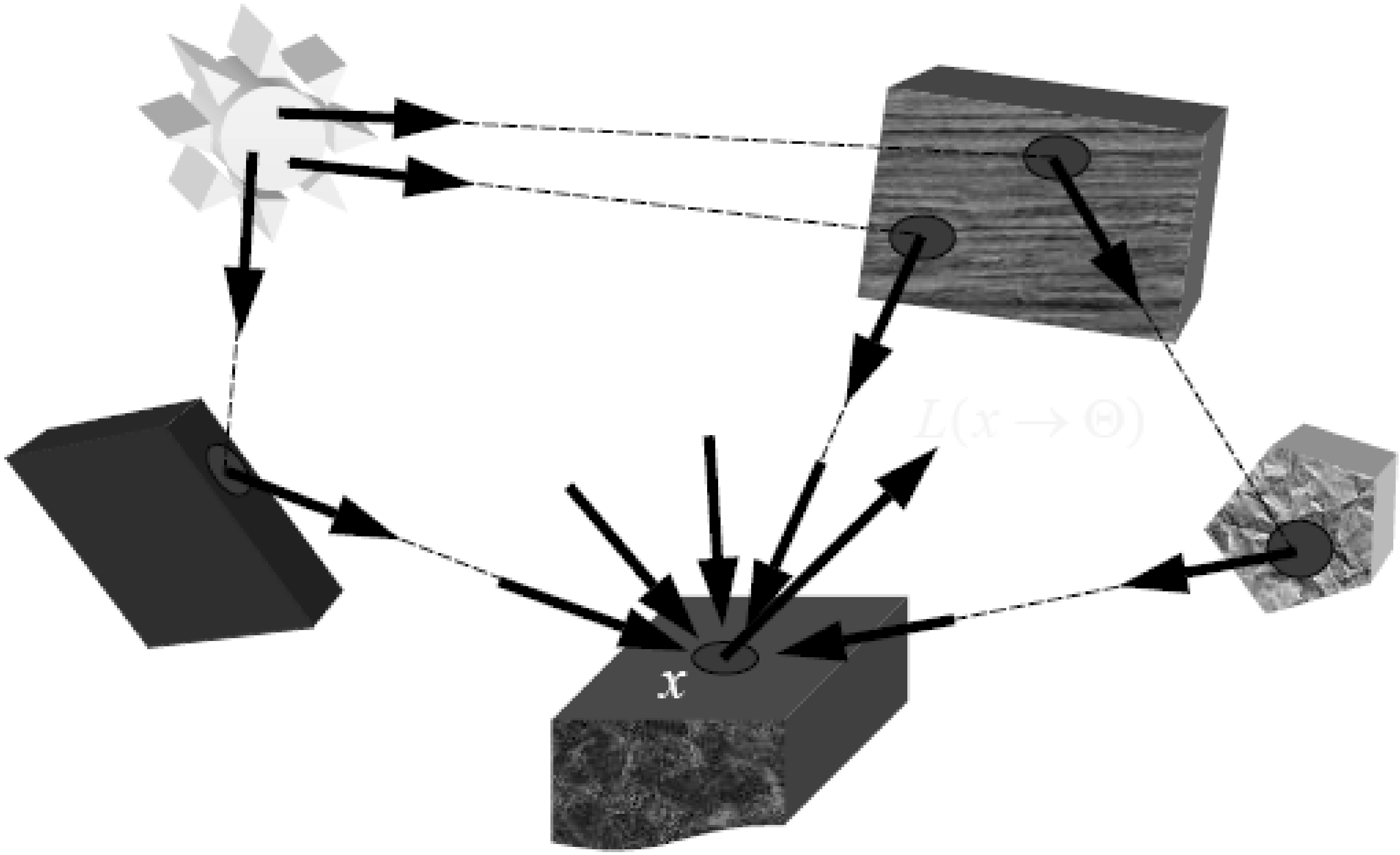
KAIST

Radiance Evaluation

- Fundamental problem in GI algorithm
  - Evaluate radiance at a given surface point in a given direction
  - Invariance defines radiance everywhere else



Radiance Evaluation

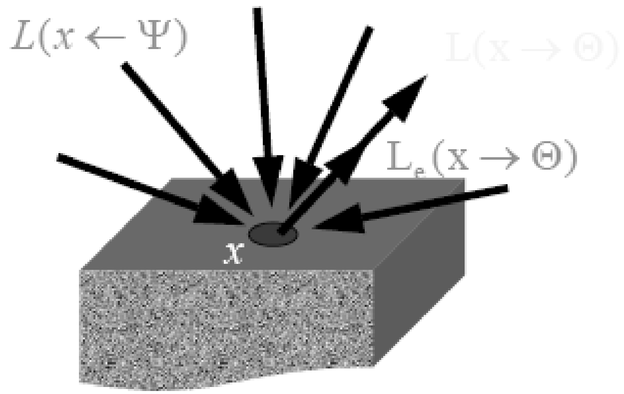


… find paths between sources and surfaces to be shaded

Why Monte Carlo?

- **Radiace is hard to evaluate**

$$L(x \rightarrow \Theta) = L_e(x \rightarrow \Theta) + \int_{\Omega_x} f_r(\Psi \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi) \cdot \cos(\Psi, n_x) \cdot d\omega_\Psi$$



From kavita's slides

- **Sample many paths**
  - **Integrate over all incoming directions**
- **Analytical integration is difficult**
  - **Need numerical techniques**

Monte Carlo Integration

- Numerical tool to evaluate integrals
- Use sampling
- Stochastic errors
- Unbiased
  - On average, we get the right answer

Probability

- Random variable x
- Possible outcomes: $x_1, x_2, x_3, \ldots, x_n$
  - each with probability: $p_1, p_2, p_3, \ldots, p_n$

- E.g. 'average die': 2,3,3,4,4,5
  - outcomes: $x_1 = 2, x_2 = 3, x_3 = 4, x_3 = 5$
  - probabilities:

$$p_1 = 1/6, p_2 = 1/3, p_3 = 1/3, p_3 = 1/6$$

Expected value

- Expected value = average value

$$E[x] = \sum_{i=1}^{n} x_i p_i$$

- E.g. die:

$$E[x] = 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{3} + 4 \cdot \frac{1}{3} + 5 \cdot \frac{1}{6} = 3.5$$

Variance

- Expected ‘distance’ to expected value

$$\sigma^2[x] = E[(x - E[x])^2]$$

- E.g. die:

$$\sigma^2[x] = (2-3.5)^2 \cdot \frac{1}{6} + (3-3.5)^2 \cdot \frac{1}{3} + (4-3.5)^2 \cdot \frac{1}{3} + (5-3.5)^2 \cdot \frac{1}{6}$$
$$= 0.916$$

- Property: $\sigma^2[x] = E[x^2] - E[x]^2$

Continuous random variable

- Random variable $x \in [a,b]$
- Probability density function (pdf) $p(x)$
- Probability that variable has value x: $p(x)dx$

$$\int_a^b p(x)dx = 1$$

- Cumulative distribution function (CDF)
  - CDF is non-decreasing, positive

$$\Pr(x \leq y) = CDF(y) = \int_{-\infty}^{y} p(x)dx$$

Continuous random variable

- Expected value: $E[x] = \int_a^b x p(x) dx$

$$E[g(x)] = \int_a^b g(x) p(x) dx$$

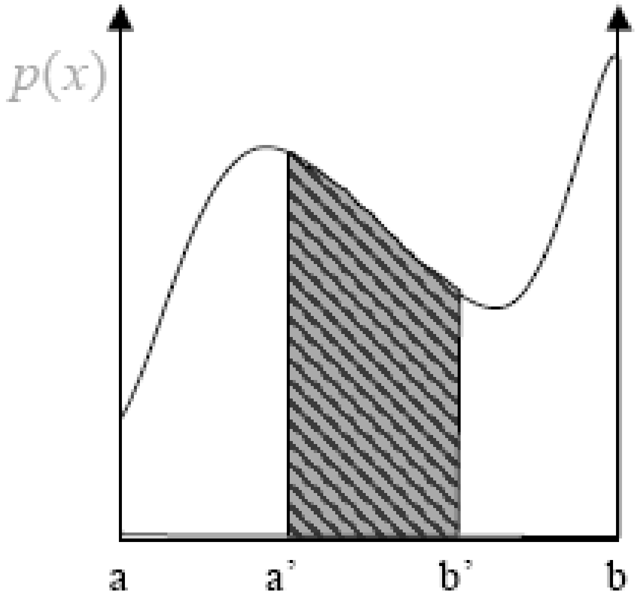
- Variance:

$$\sigma^2[x] = \int_a^b (x - E[x])^2 p(x) dx$$

$$\sigma^2[g(x)] = \int_a^b (g(x) - E[g(x)])^2 p(x) dx$$

- Deviation: $\sigma[x], \sigma[g(x)]$

Continuous random variable



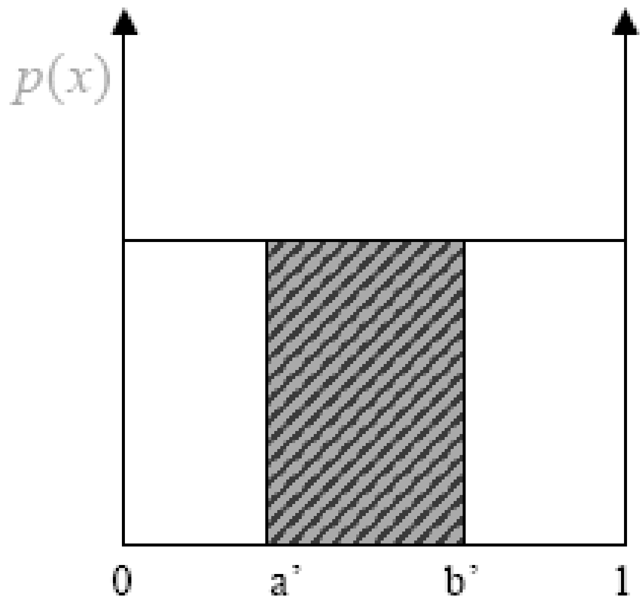
$$\int_a^b p(x)dx = 1$$

$$\Pr(x \leq y) = CDF(y) = \int_{-\infty}^{y} p(x)dx$$

Probability that x belongs to [a',b'] $= \Pr(x \leq b') - \Pr(x \leq a')$

$$= \int_{-\infty}^{b'} p(x)dx - \int_{-\infty}^{a'} p(x)dx = \int_{a'}^{b'} p(x)dx$$

Uniform distribution



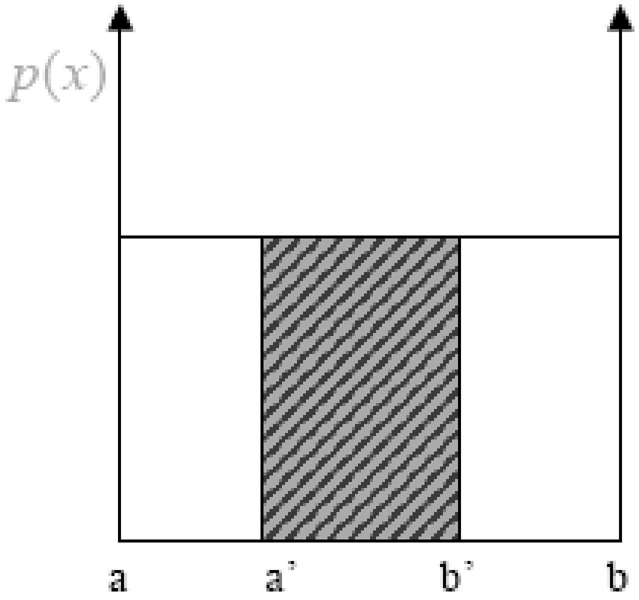
$$\int_a^b p(x)dx = 1$$

$$p(x) = \frac{1}{1-0} = 1$$

$$\Pr(x \in [a', b']) = \int_{a'}^{b'} 1dx = (b' - a')$$

$$\Pr(x \le y) = CDF(y) = \int_{-\infty}^{y} p(x)dx = y$$

Uniform distribution



$$\int_a^b p(x)dx = 1$$

$$p(x) = \frac{1}{b-a}$$

Probability that x belongs to [a',b'] $= \int_{a'}^{b'} \frac{1}{(b-a)} dx = \frac{(b'-a')}{(b-a)}$

$$\Pr(x \leq y) = CDF(y) = \int_{-\infty}^{y} p(x)dx = \frac{(y-a)}{(b-a)}$$

More than one sample

- Consider the weighted sum of N samples

- Expected value $E[\frac{1}{N}(x^1 + x^2 + x^3 + \dots x^N)] = E[x]$

- Variance $\sigma^2[\frac{1}{N}(x^1 + x^2 + x^3 + \dots x^N)] = \frac{1}{N}\sigma^2[x]$

- Deviation $\sigma[\frac{1}{N}(x^1 + x^2 + x^3 + \dots x^N)] = \frac{1}{\sqrt{N}}\sigma[x]$

More than one sample

- Consider the weighted sum of N samples

$$g(x) = \frac{1}{N}(f(x_1) + f(x_2) + f(x_3) + \ldots f(x_N))$$

- Expected value

$$E[g(x)] = E[\frac{1}{N}\sum_i^N f(x_i)] = E[f(x)]$$

- Variance

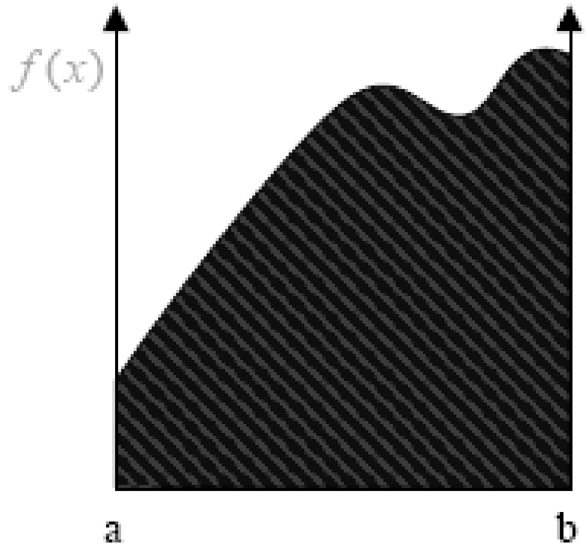
$$\sigma^2[g(x)] = \sigma^2[\frac{1}{N}\sum_i^N f(x_i)] = \frac{1}{N}\sigma^2[f(x)]$$

- Deviation $\sigma[g(x)] = \frac{1}{\sqrt{N}}\sigma[f(x)]$

Numerical Integration

- A one-dimensional integral:

$$I = \int_a^b f(x)dx$$

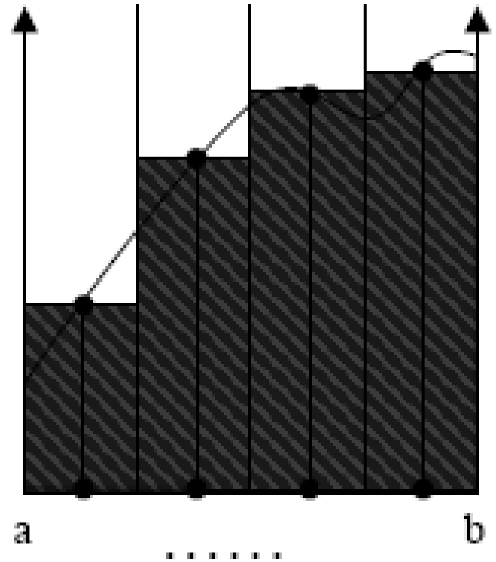


Deterministic Integration

- Quadrature rules:

$$I = \int_a^b f(x)dx$$

$$\approx \sum_{i=1}^{N} w_i f(x_i)$$

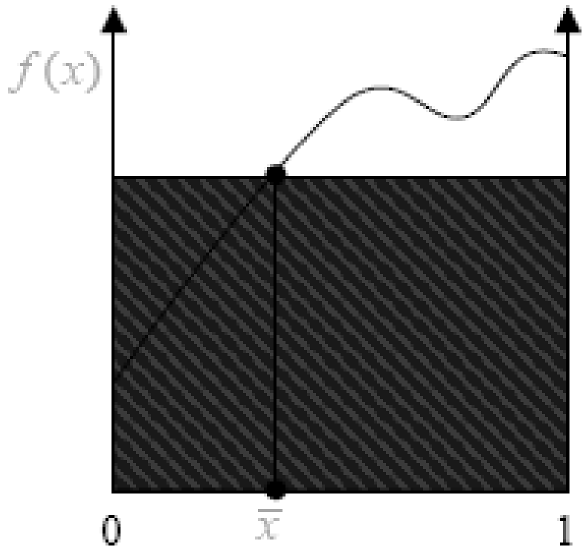


Monte Carlo Integration

Primary estimator:

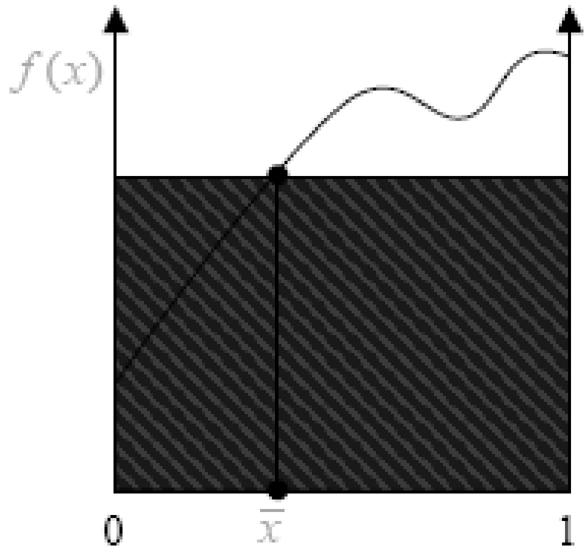
$$I = \int_a^b f(x)dx$$

$$I_{prim} = f(\bar{x})$$



Monte Carlo Integration

Primary estimator:



$$I = \int_a^b f(x)dx$$

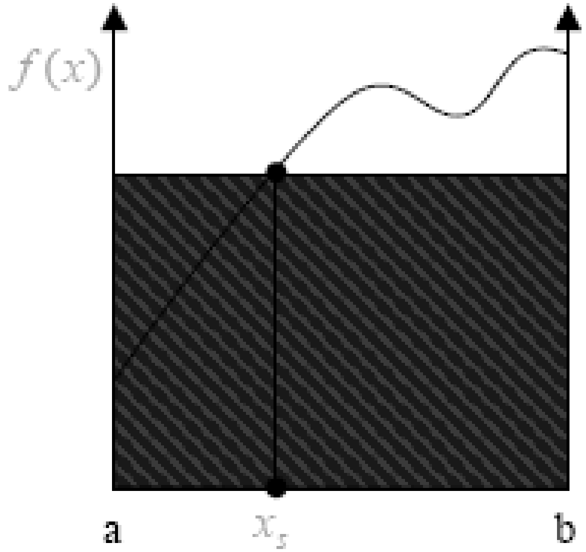
$$I_{prim} = f(\bar{x})$$

$$E(I_{prim}) = \int_0^1 f(x)\,p(x)dx = \int_0^1 f(x)1dx = I$$

Unbiased estimator!

Monte Carlo Integration

Primary estimator:



$$I = \int_a^b f(x)dx$$

$$I_{prim} = f(x_s)(b-a)$$

$$E(I_{prim}) = \int_a^b f(x)(b-a)\,p(x)dx = \int_a^b f(x)(b-a)\frac{1}{(b-a)}dx = I$$

Unbiased estimator!

Monte Carlo Integration: Error

Variance of the estimator $\rightarrow$ a measure of the stochastic error

$$\sigma_{prim}^2 = \int_a^b [\frac{f(x)}{p(x)} - I]^2 p(x)dx$$

- **Consider p(x) for estimate**
- **We will study it as importance sampling later**

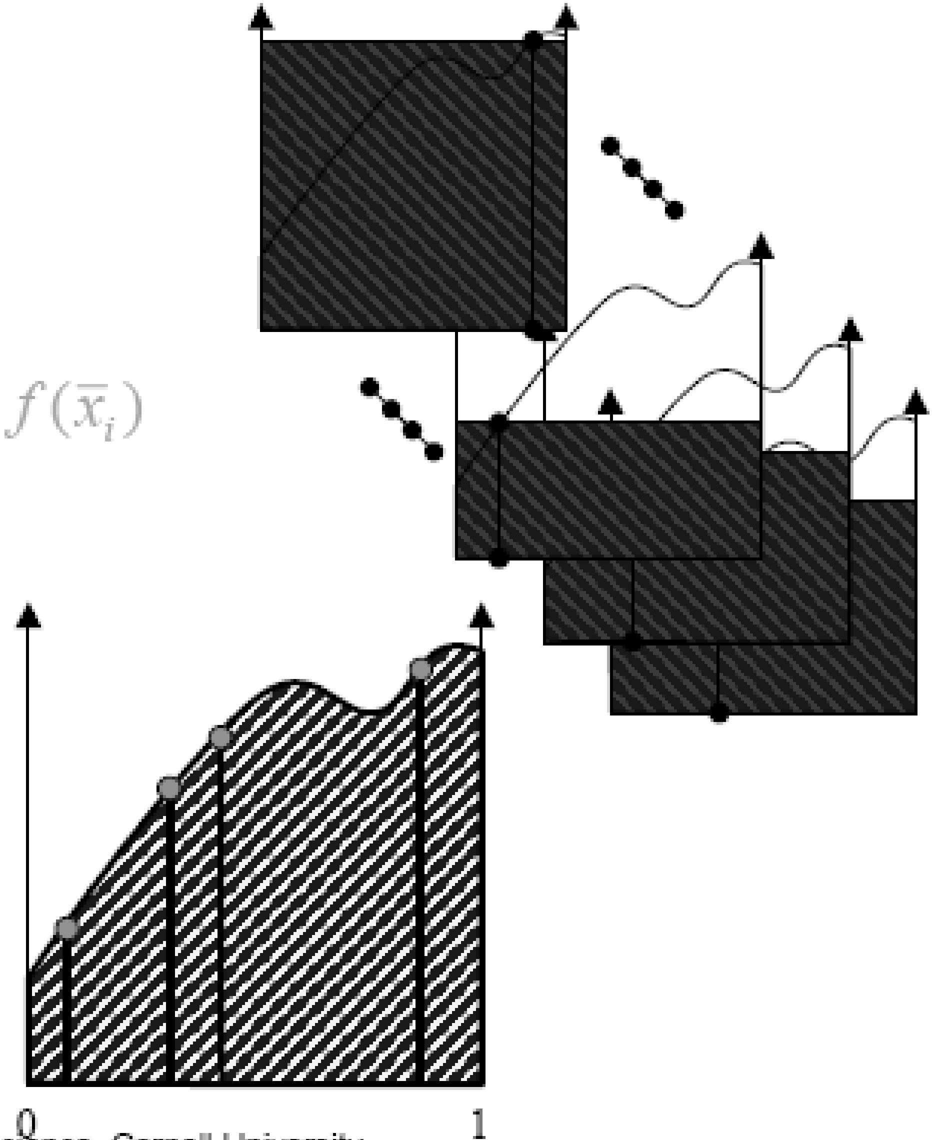
More samples

Secondary estimator

Generate N random samples x_i

Estimator: $\langle I \rangle = I_{sec} = \frac{1}{N}\sum_{i=1}^{N} f(\bar{x}_i)$

Variance $\sigma_{sec}^2 = \sigma_{prim}^2 / N$



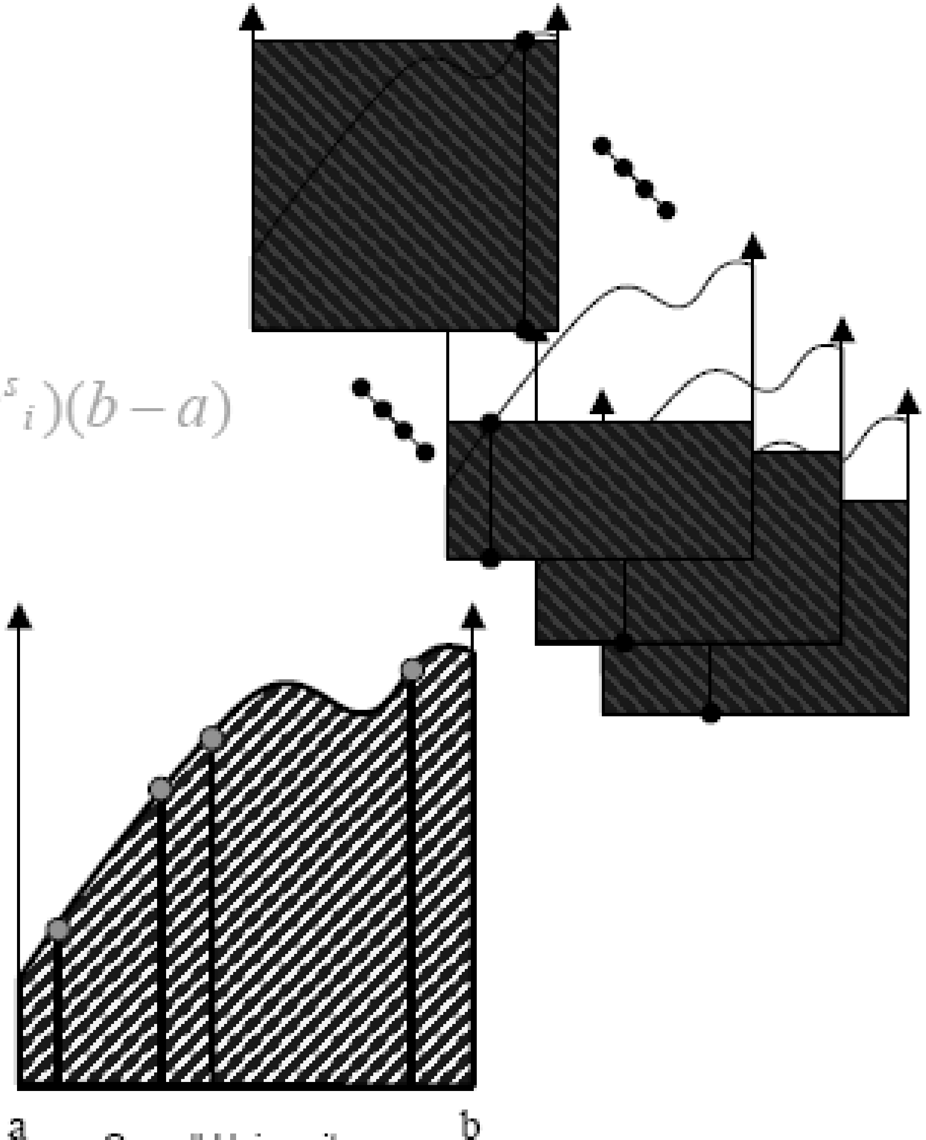
More samples

Secondary estimator

Generate N random samples x_i

Estimator: $\langle I \rangle = I_{\text{sec}} = \frac{1}{N}\sum_{i=1}^{N} f(x^s{}_i)(b-a)$

Variance $\sigma^2_{\text{sec}} = \sigma^2_{prim} / N$



Monte Carlo Integration

- Expected value of estimator

$$E[\langle I \rangle] = E[\frac{1}{N}\sum_i^N \frac{f(x_i)}{p(x_i)}] = \frac{1}{N}\int (\sum_i^N \frac{f(x_i)}{p(x_i)}) p(x)dx$$

$$= \frac{1}{N}\sum_i^N \int (\frac{f(x)}{p(x)}) p(x)dx$$

$$= \frac{N}{N}\int f(x)dx = I$$

  – on ‘average’ get right result: **unbiased**

- Standard deviation σ is a measure of the stochastic error

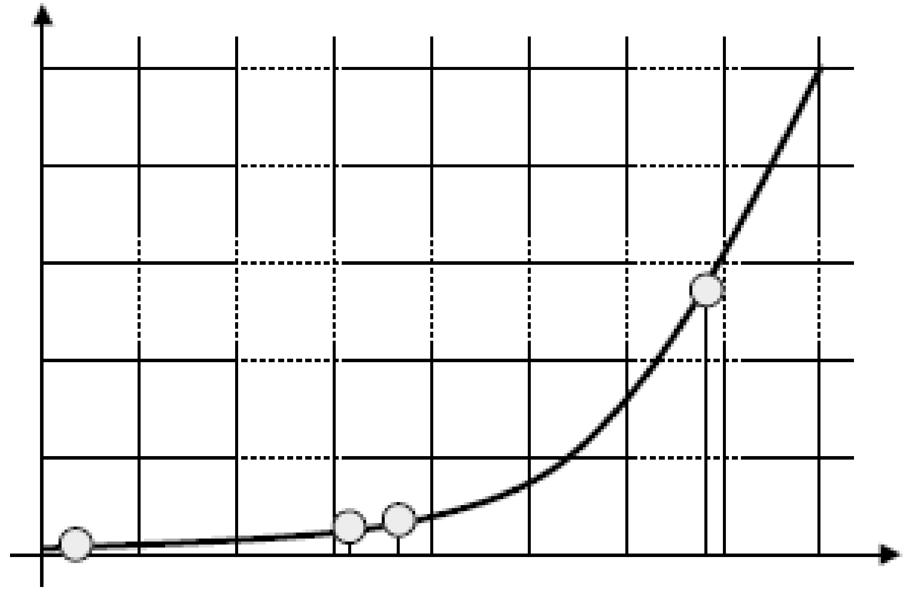
$$\sigma^2 = \frac{1}{N}\int_a^b [\frac{f(x)}{p(x)} - I]^2 p(x)dx$$

MC Integration - Example

- Integral $I = \int_0^1 5x^4 dx = 1$

- Uniform sampling

- Samples :



$x_1 = .86 \qquad \langle I \rangle = 2.74$

$x_2 = .41 \qquad \langle I \rangle = 1.44$

$x_3 = .02 \qquad \langle I \rangle = 0.96$

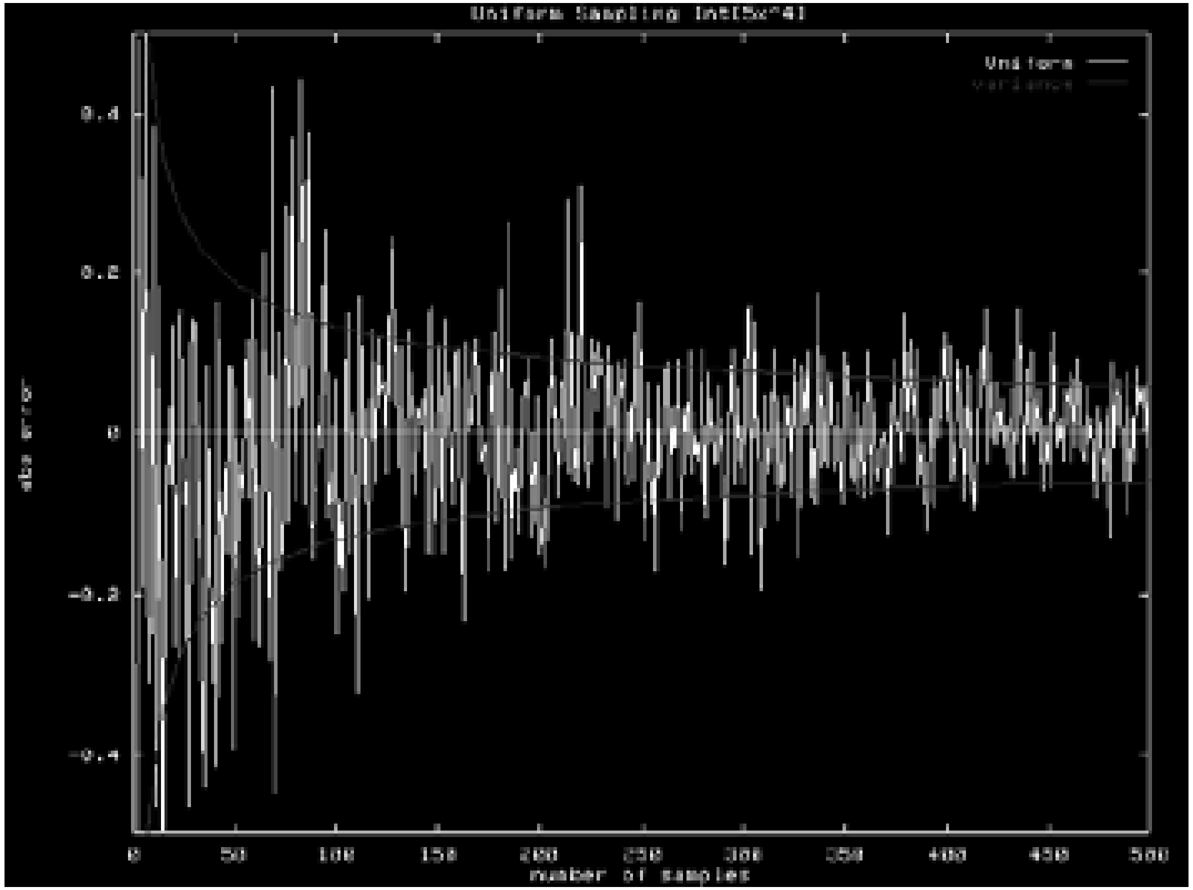
$x_4 = .38 \qquad \langle I \rangle = 0.75$

MC Integration - Example

- **Integral**

$$I = \int_0^1 5x^4 dx = 1$$

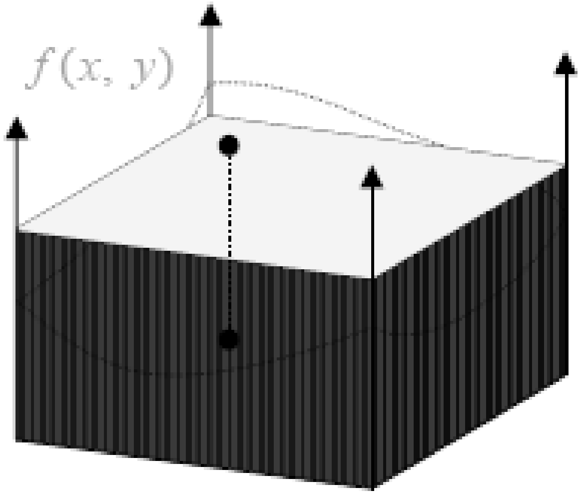
- **Variance**



MC Integration: 2D

- Primary estimator:

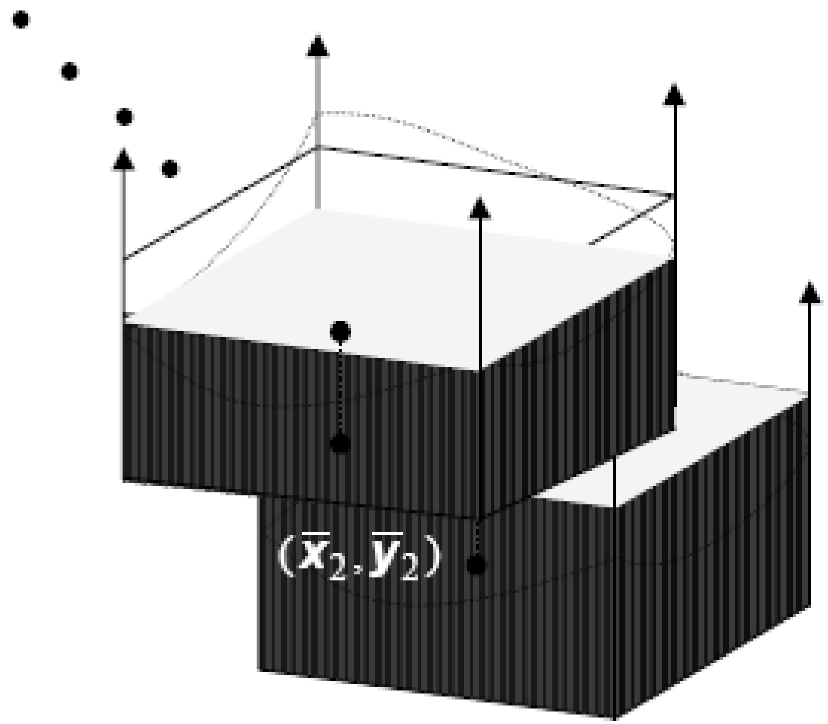
$$\bar{I}_{prim} = \frac{f(\bar{x}, \bar{y})}{p(\bar{x}, \bar{y})}$$



MC Integration: 2D

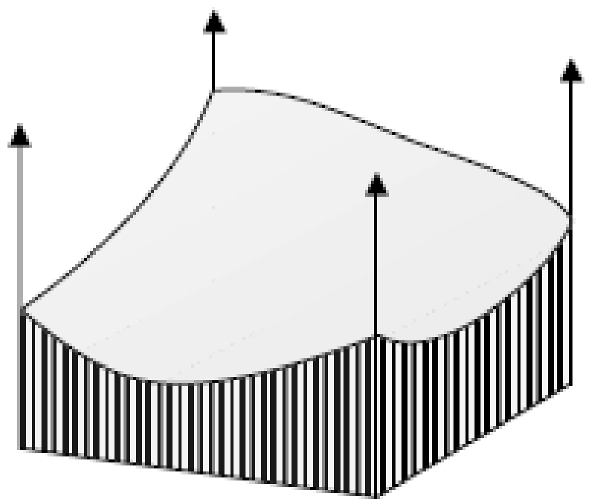
- Secondary estimator:

$$I_{\text{sec}} = \frac{1}{N}\sum_{i=1}^{N}\frac{f(\bar{x}_i,\bar{y}_i)}{p(\bar{x}_i,\bar{y}_i)}$$



Monte Carlo Integration - 2D

- MC Integration works well for higher dimensions
- Unlike quadrature



$$I = \int_a^b \int_c^d f(x, y)\,dx\,dy$$

$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i, y_i)}{p(x_i, y_i)}$$

Advantages of MC

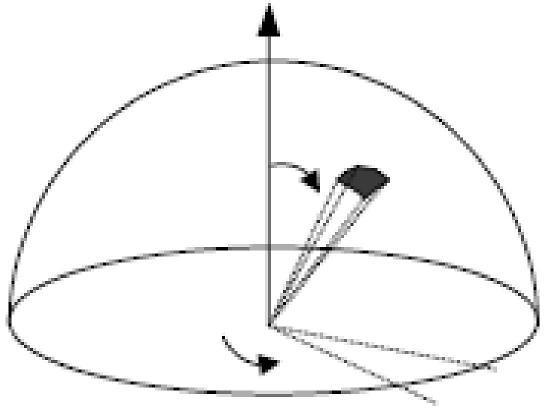
- **Convergence rate of** $O(\frac{1}{\sqrt{N}})$
- **Simple**
  - **Sampling**
  - **Point evaluation**
- **General**
  - **Works for high dimensions**
  - **Deals with discontinuities, crazy functions, etc.**

MC Integration - 2D example

- Integration over hemisphere:

$$I = \int_{\Omega} f(\Theta) d\omega_{\Theta}$$

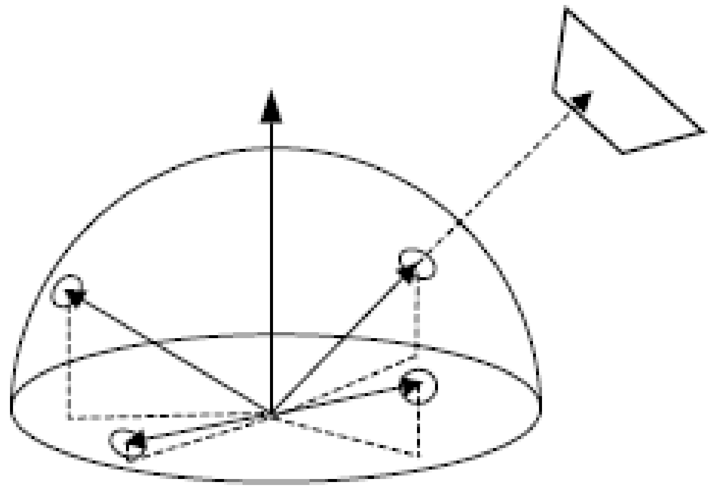
$$= \int_{0}^{2\pi} \int_{0}^{\pi/2} f(\varphi, \theta) \sin\theta \, d\theta \, d\varphi$$



$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(\varphi_i, \theta_i) \sin\theta}{p(\varphi_i, \theta_i)}$$

Hemisphere Integration example

Irradiance due to light source:



$$I = \int_{\Omega} L_{source} \cos\theta d\omega_{\Theta}$$

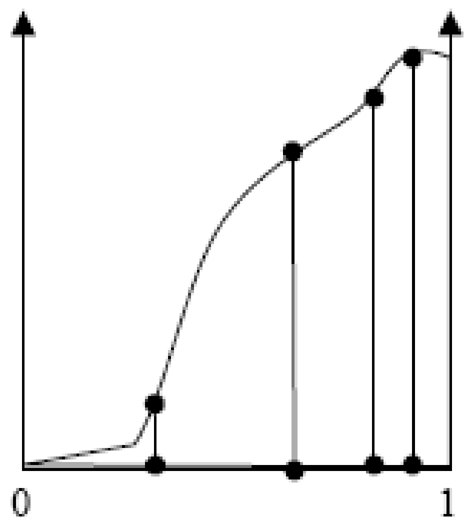
$$= \int_0^{2\pi}\int_0^{\pi/2} L_{source} \cos\theta \sin\theta d\theta d\varphi$$

$$p(\omega_i) = \frac{\cos\theta \sin\theta}{\pi}$$

$$\langle I \rangle = \frac{1}{N}\sum_{i=1}^{N} \frac{L_{source}(\omega_i)\cos\theta \sin\theta}{p(\omega_i)} = \frac{\pi}{N}\sum_{i=1}^{N} L_{source}(\omega_i)$$

Importance Sampling

- **Take more samples in important regions, where the function is large**



From kavita's slides

MC integration - Non-Uniform

- Some parts of the integration domain have higher importance
- Generate samples according to density function p(x)

$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)}$$

- Estimator?
- What is optimal p(x)? $p(x) \approx f(x) / \int f(x) dx$

MC integration - Non-Uniform

- Generate samples according to density function p(x)

$$p(x) \approx f(x) / \int f(x)dx$$

- Why?

$$I_{estimator} = \frac{1}{N}\sum\frac{f(x)}{p(x)} = \frac{1}{N}\sum\frac{f(x)}{f(x)/I} = \frac{1}{N}\sum I = I$$

- But…..

$$\sigma^2 = \frac{1}{N}\int_a^b [\frac{f(x)}{p(x)} - I]^2 p(x)dx$$

$$= \frac{1}{N}\int_a^b [\frac{f(x)}{f(x)/I} - I]^2 p(x)dx = 0$$

Example

- Function: $I = \int_0^4 x dx = 8$ $\qquad f(x) = x$

$$\sigma^2 = \frac{1}{N}\int_a^b [\frac{f(x)}{p(x)} - I]^2 p(x) dx$$

$$p(x) = \frac{x}{8}, \sigma^2 = 0 \qquad I_{estimator} = I = 8$$

$$p(x) = \frac{1}{4}, \sigma^2 = \frac{1}{N}\int_0^4 [\frac{x}{1/4} - 8]^2 \frac{1}{4} dx = 21.3 / N$$

$$p(x) = \frac{x+2}{16}, \sigma^2 = \frac{1}{N}\int_0^4 [\frac{x}{(x+2)/16} - 8]^2 \frac{x+2}{16} dx = 6.3 / N$$

Importance Sampling

- Generate samples from density function p(x)

$$\langle I \rangle = \frac{1}{N}\sum_{i=1}^{N}\frac{f(x_i)}{p(x_i)}$$

- Optimal p(x)? $p(x) \approx f(x) / \int f(x)dx$

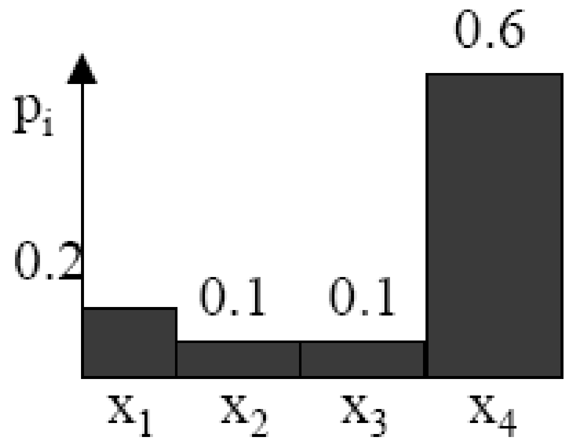
- General principle:
  - Closer shape of p(x) is to shape of f(x), lower the variance
- Variance can *increase* if p(x) is chosen badly

Sampling according to pdf

- Inverse cumulative distribution function
- Rejection sampling

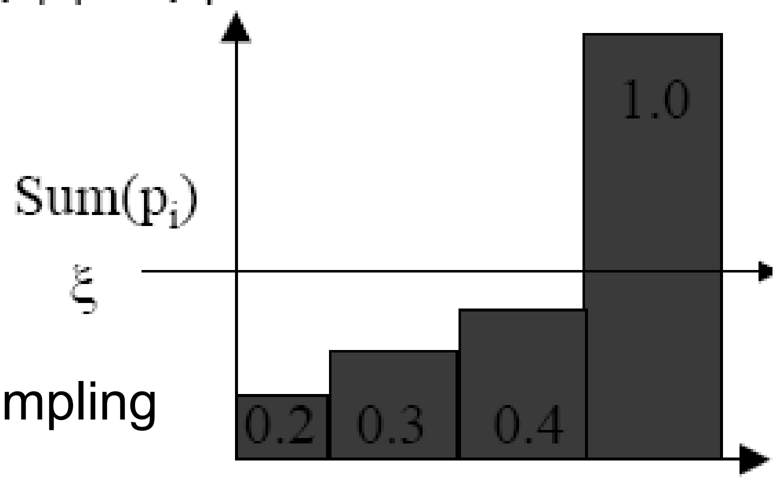
Inverse Cumulative Distribution Function – Discrete Case

- Consider discrete events x_i
  - with probability p_i



- Select x_i if:

$$p_1 + \ldots p_{i-1} < \xi < p_1 + \ldots p_{i-1} + p_i$$



$$\sum_{j=1}^{i-1} p_j < \xi < \sum_{j=1}^{i} p_j$$

$$P(x_i) = P(\xi \in [\sum_{j=1}^{i-1} p_j, \sum_{j=1}^{i} p_j])$$

$P(a < \xi < b) = (b - a)$, given uniform sampling

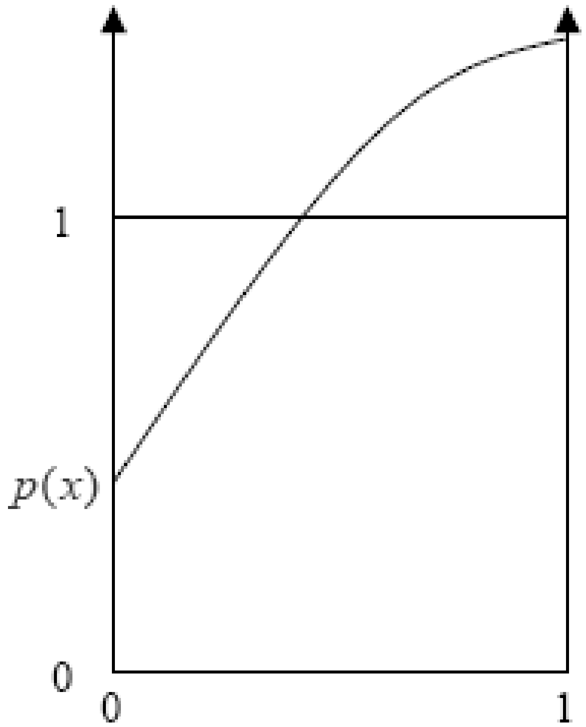
$$P(x_i) = P(\xi \in [\sum_{j=1}^{i-1} p_j, \sum_{j=1}^{i} p_j]) = \sum_{j=1}^{i} p_j - \sum_{j=1}^{i-1} p_j = p_i$$

Continuous Random Variable

- **Algorithm**
  - **Pick u uniformly from [0, 1)**
  - **Output y = P^{-1}(u), where** $P(y) = \int_{-\infty}^{y} p(x)dx$

Non-Uniform Samples

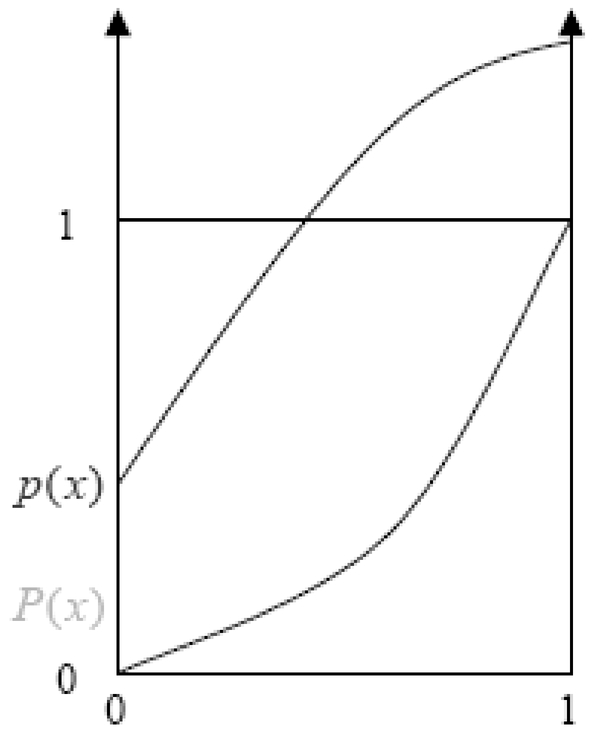
- 1) Choose a normalized probability density function *p(x)*



Non-Uniform Samples

- 1) Choose a normalized probability density function *p(x)*
- 2) Integrate to get a cumulative probability distribution function *P(x)*:

$$P(x) = \int_0^x p(t)dt$$



Note this is similar to computing $\sum_{j=1}^{i} p_j$

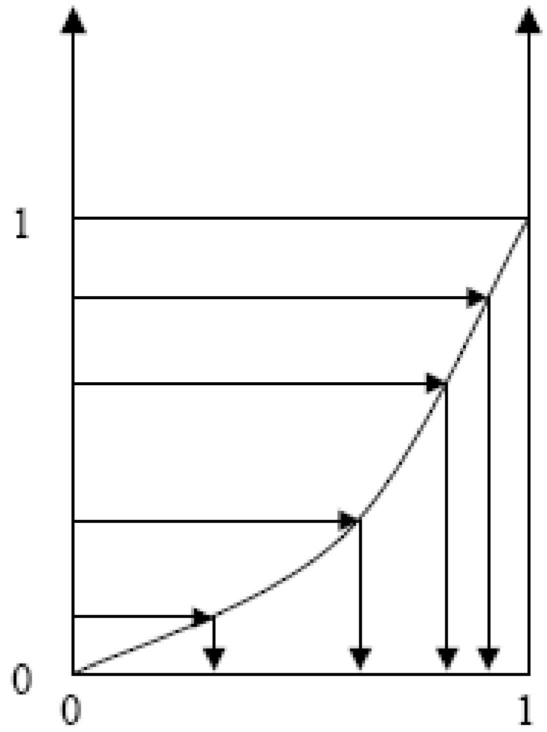
Non-Uniform Samples

- 1) Choose a normalized probability density function *p(x)*
- 2) Integrate to get a probability distribution function *P(x)*:

$$P(x) = \int_0^x p(t)dt$$

- 3) Invert *P*:

$$x = P^{-1}(\xi)$$



Note this is similar to going from y axis to x in discrete case!

Cosine distribution

$$f = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \cos\theta \sin\theta \, d\theta \, d\phi$$

$$p(\theta, \phi) = \frac{\cos\theta \sin\theta}{\pi}$$

$$CDF(\theta, \phi) = \int_0^{\theta} \int_0^{\phi} \frac{\cos\theta \sin\theta}{\pi} d\theta \, d\phi = (1 - \cos^2\theta) \frac{\phi}{2\pi}$$

$$F(\theta) = 1 - \cos^2\theta$$

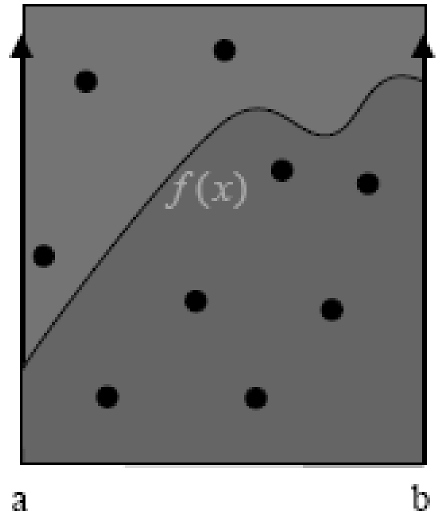
$$F(\phi) = \frac{\phi}{2\pi}$$

$$\phi_i = 2\pi\xi_1 \qquad \theta_i = \cos^{-1}\sqrt{\xi_2}$$

Rejection Method

- **Often not possible to compute the inverse of cdf**
  - Pick ξ_1, ξ_2

$$I = \int_a^b f(x)dx$$



From kavita's slides

  - If $\xi_2 < f(\xi_1)$, select ξ_1

  - Is this efficient? What determines efficiency? A(f)/A(rectangle)

Class Objectives were:

- Sampling approach for solving the rendering equation
  - Monte Carlo integration
  - Estimator and its variance
  - Sampling according to the pdf

Next Time

- Monte Carlo ray tracing