
CS580:
Monte Carlo Integration

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Course URL:
<http://sglab.kaist.ac.kr/~sungeui/GCG>

KAIST



Class Objectives

- **Sampling approach for solving the rendering equation**
 - Monte Carlo integration
 - Estimator and its variance
 - Sampling according to the pdf

- **Refer to the chapter of Monte Carlo Integration of my book**

Two Forms of the Rendering Equation

- Hemisphere integration

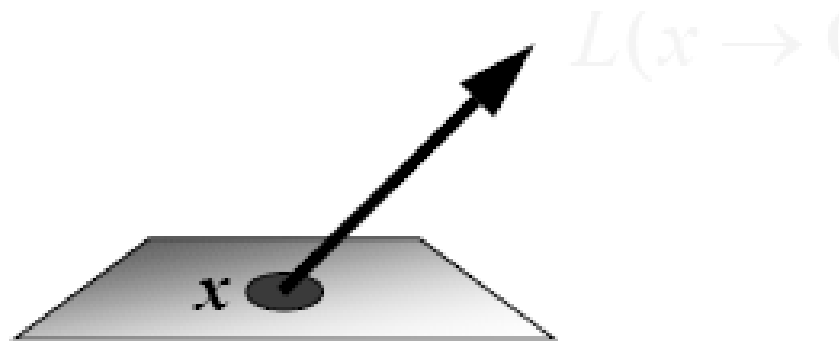
$$L(x \rightarrow \Theta) = L_e(x \rightarrow \Theta) + \int_{\Omega_x} f_r(\Psi \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi) \cdot \cos \theta_x \cdot d\omega_\Psi$$

- Area integration

$$L(x \rightarrow \Theta) = L_e(x \rightarrow \Theta) + \int_A f_r(\Psi \leftrightarrow \Theta) \cdot L(y \rightarrow -\Psi) \cdot \frac{\cos \theta_x \cdot \cos \theta_y}{r_{xy}^2} \cdot V(x, y) \cdot dA_y$$

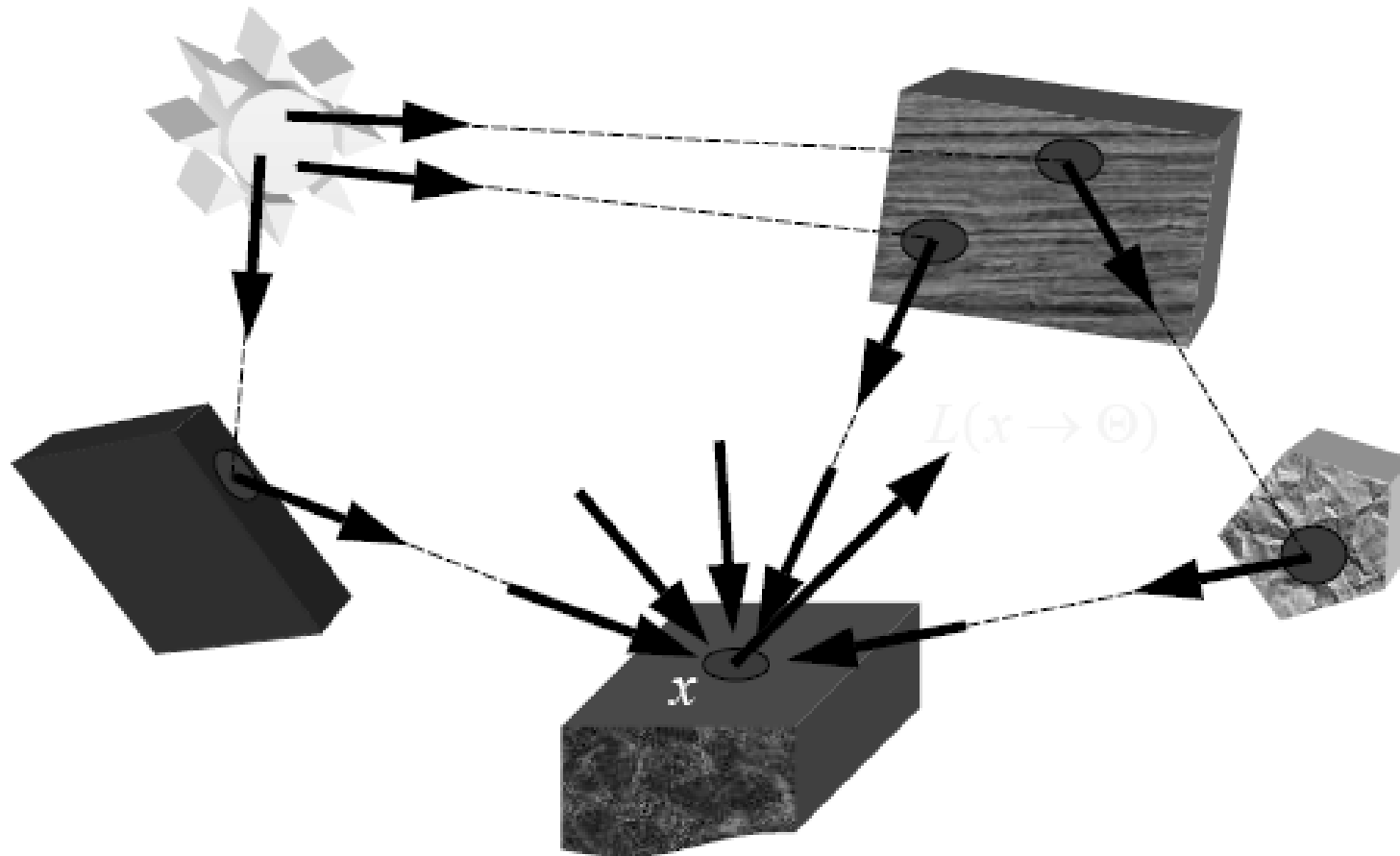
Radiance Evaluation

- **Fundamental problem in GI algorithm**
 - Evaluate radiance at a given surface point in a given direction
 - Invariance defines radiance everywhere else



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Radiance Evaluation

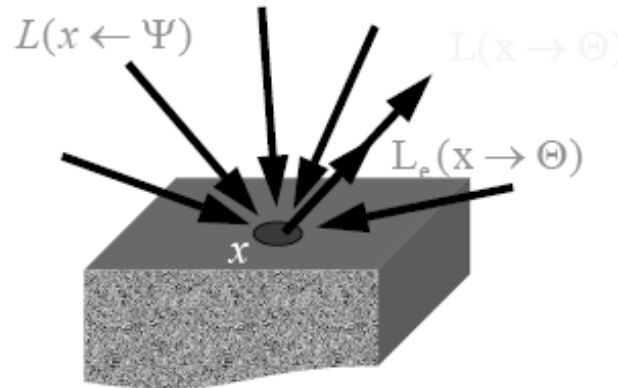


... find paths between sources and surfaces to be shaded

Why Monte Carlo?

- Radiance is hard to evaluate

$$\underline{L(x \rightarrow \Theta)} = \underline{L_e(x \rightarrow \Theta)} + \int_{\Omega_x} \underline{f_r(\Psi \leftrightarrow \Theta)} \cdot \underline{L(x \leftarrow \Psi)} \cdot \cos(\Psi, n_x) \cdot d\omega_\Psi$$



From kavita's slides

- Sample many paths
 - Integrate over all incoming directions
- Analytical integration is difficult
 - Need numerical techniques

Monte Carlo Integration

- Numerical tool to evaluate integrals
- Use sampling
- Stochastic errors
- Unbiased
 - On average, we get the right answer

Probability

- Random variable x
- Possible outcomes: $x_1, x_2, x_3, \dots, x_n$
 - each with probability: $p_1, p_2, p_3, \dots, p_n$
- E.g. ‘average die’: 2, 3, 3, 4, 4, 5
 - outcomes: $x_1 = 2, x_2 = 3, x_3 = 4, x_4 = 5$
 - probabilities:

$$p_1 = 1/6, p_2 = 1/3, p_3 = 1/3, p_4 = 1/6$$

Expected value

- Expected value = average value

$$E[x] = \sum_{i=1}^n x_i p_i$$

- E.g. die:

$$E[x] = 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{3} + 4 \cdot \frac{1}{3} + 5 \cdot \frac{1}{6} = 3.5$$

Variance

- Expected 'distance' to expected value

$$\sigma^2[x] = E[(x - E[x])^2]$$

- E.g. die:

$$\begin{aligned}\sigma^2[x] &= (2 - 3.5)^2 \cdot \frac{1}{6} + (3 - 3.5)^2 \cdot \frac{1}{3} + (4 - 3.5)^2 \cdot \frac{1}{3} + (5 - 3.5)^2 \cdot \frac{1}{6} \\ &= 0.916\end{aligned}$$

- Property: $\sigma^2[x] = E[x^2] - E[x]^2$

Continuous random variable

- Random variable $x \in [a, b]$
- Probability density function (pdf) $p(x)$
- Probability that variable has value x : $p(x)dx$

$$\int_a^b p(x)dx = 1$$

- Cumulative distribution function (CDF)
 - CDF is non-decreasing, positive

$$\Pr(x \leq y) = CDF(y) = \int_{-\infty}^y p(x)dx$$

Continuous random variable

- Expected value: $E[x] = \int_a^b xp(x)dx$

$$E[g(x)] = \int_a^b g(x)p(x)dx$$

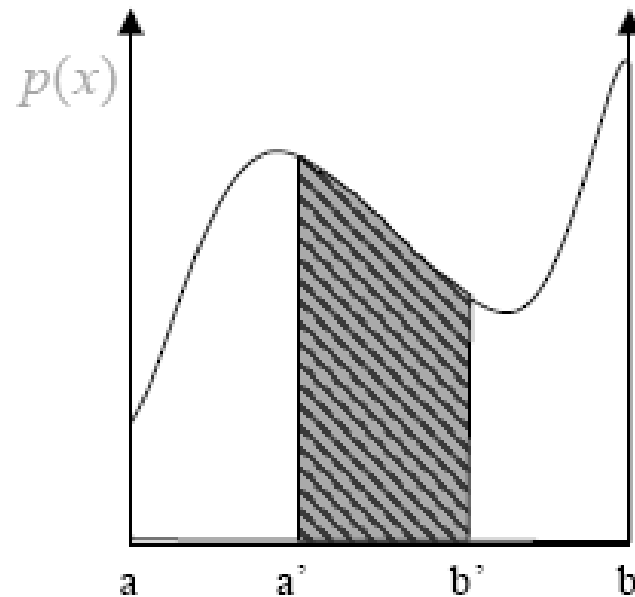
- Variance:

$$\sigma^2[x] = \int_a^b (x - E[x])^2 p(x)dx$$

$$\sigma^2[g(x)] = \int_a^b (g(x) - E[g(x)])^2 p(x)dx$$

- Deviation: $\sigma[x], \sigma[g(x)]$

Continuous random variable



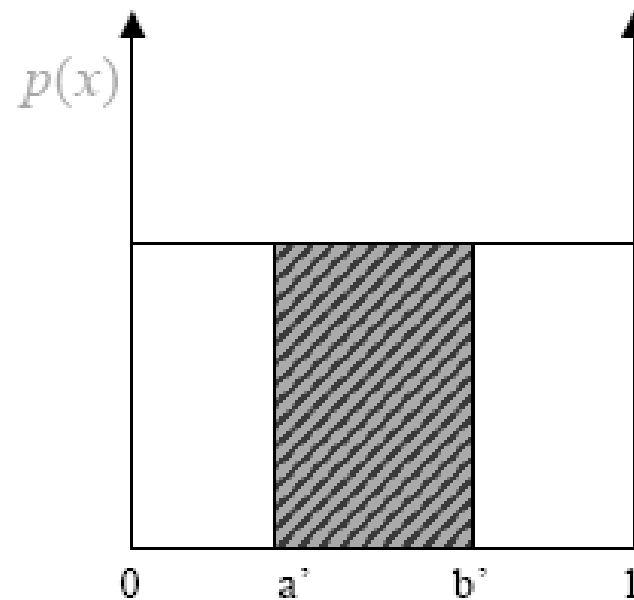
$$\int_a^b p(x) dx = 1$$

$$\Pr(x \leq y) = CDF(y) = \int_{-\infty}^y p(x) dx$$

Probability that x belongs to $[a', b'] = \Pr(x \leq b') - \Pr(x \leq a')$

$$= \int_{-\infty}^{b'} p(x) dx - \int_{-\infty}^{a'} p(x) dx = \int_{a'}^{b'} p(x) dx$$

Uniform distribution



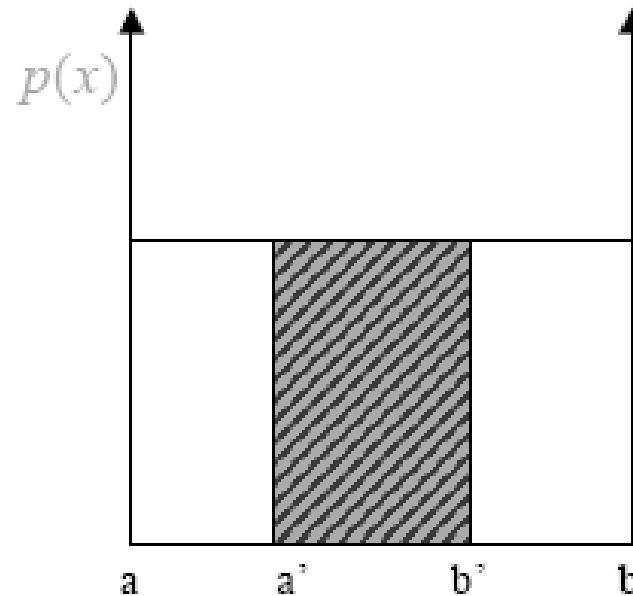
$$\int_a^b p(x) dx = 1$$

$$p(x) = \frac{1}{1-0} = 1$$

$$\Pr(x \in [a', b']) = \int_{a'}^{b'} 1 dx = (b' - a')$$

$$\Pr(x \leq y) = CDF(y) = \int_{-\infty}^y p(x) dx = y$$

Uniform distribution



$$\int_a^b p(x) dx = 1$$

$$p(x) = \frac{1}{b-a}$$

Probability that x belongs to $[a', b'] = \int_{a'}^{b'} \frac{1}{(b-a)} dx = \frac{(b'-a')}{(b-a)}$

$$\Pr(x \leq y) = CDF(y) = \int_{-\infty}^y p(x) dx = \frac{(y-a)}{(b-a)}$$

More than one sample

- Consider the weighted sum of N samples
- Expected value $E\left[\frac{1}{N}(x^1 + x^2 + x^3 + \dots x^N)\right] = E[x]$
- Variance $\sigma^2\left[\frac{1}{N}(x^1 + x^2 + x^3 + \dots x^N)\right] = \frac{1}{N}\sigma^2[x]$
- Deviation $\sigma\left[\frac{1}{N}(x^1 + x^2 + x^3 + \dots x^N)\right] = \frac{1}{\sqrt{N}}\sigma[x]$

More than one sample

- Consider the weighted sum of N samples

$$g(x) = \frac{1}{N} (f(x_1) + f(x_2) + f(x_3) + \dots + f(x_N))$$

- Expected value

$$E[g(x)] = E\left[\frac{1}{N} \sum_i^N f(x_i)\right] = E[f(x)]$$

- Variance

$$\sigma^2[g(x)] = \sigma^2\left[\frac{1}{N} \sum_i^N f(x_i)\right] = \frac{1}{N} \sigma^2[f(x)]$$

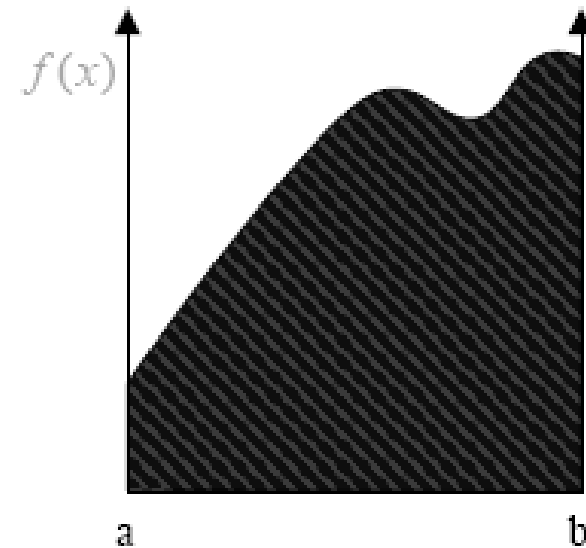
- Deviation

$$\sigma[g(x)] = \frac{1}{\sqrt{N}} \sigma[f(x)]$$

Numerical Integration

- A one-dimensional integral:

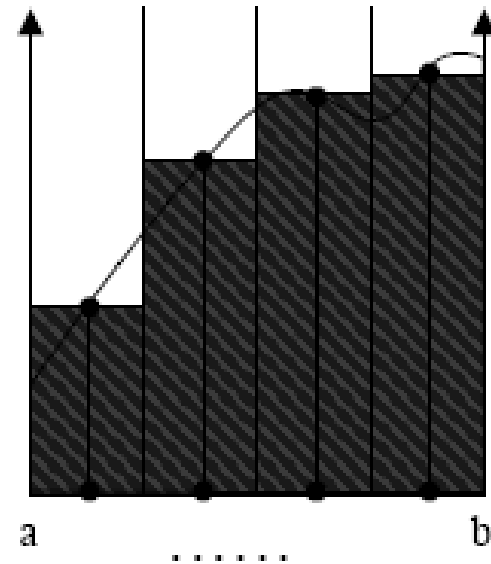
$$I = \int_a^b f(x) dx$$



Deterministic Integration

- Quadrature rules:

$$I = \int_a^b f(x) dx$$
$$\approx \sum_{i=1}^N w_i f(x_i)$$

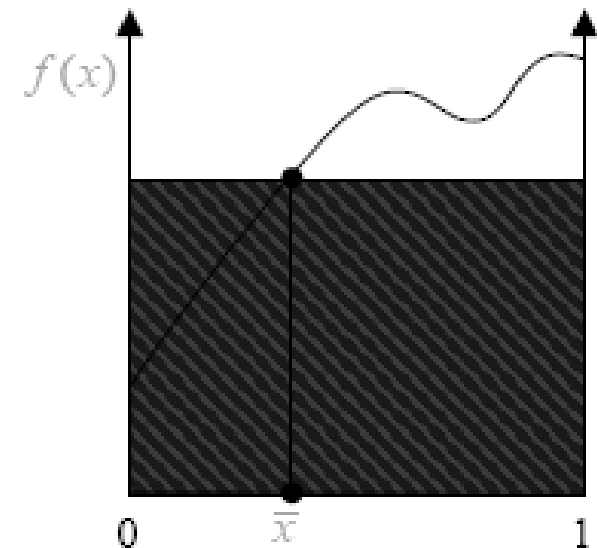


Monte Carlo Integration

Primary estimator:

$$I = \int_a^b f(x) dx$$

$$I_{prim} = f(\bar{x})$$

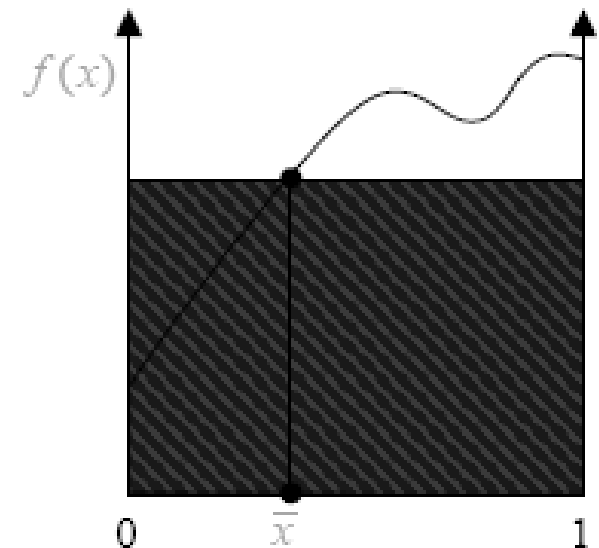


Monte Carlo Integration

Primary estimator:

$$I = \int_a^b f(x) dx$$

$$I_{prim} = f(\bar{x})$$



$$E(I_{prim}) = \int_0^1 f(x) p(x) dx = \int_0^1 f(x) 1 dx = I$$

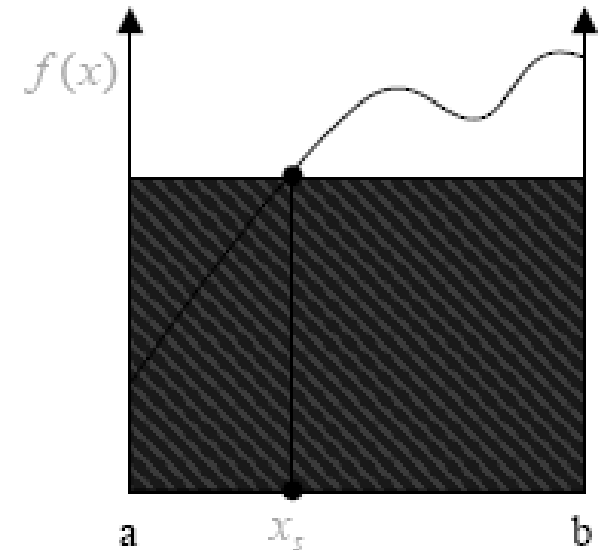
Unbiased estimator!

Monte Carlo Integration

Primary estimator:

$$I = \int_a^b f(x) dx$$

$$I_{prim} = f(x_s)(b - a)$$



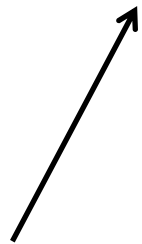
$$E(I_{prim}) = \int_a^b f(x)(b - a)p(x) dx = \int_a^b f(x)(b - a) \frac{1}{(b - a)} dx = I$$

Unbiased estimator!

Monte Carlo Integration: Error

Variance of the estimator → a measure of the stochastic error

$$\sigma_{prim}^2 = \int_a^b \left[\frac{f(x)}{p(x)} - I \right]^2 p(x) dx$$



- Consider $p(x)$ for estimate
- We will study it as importance sampling later

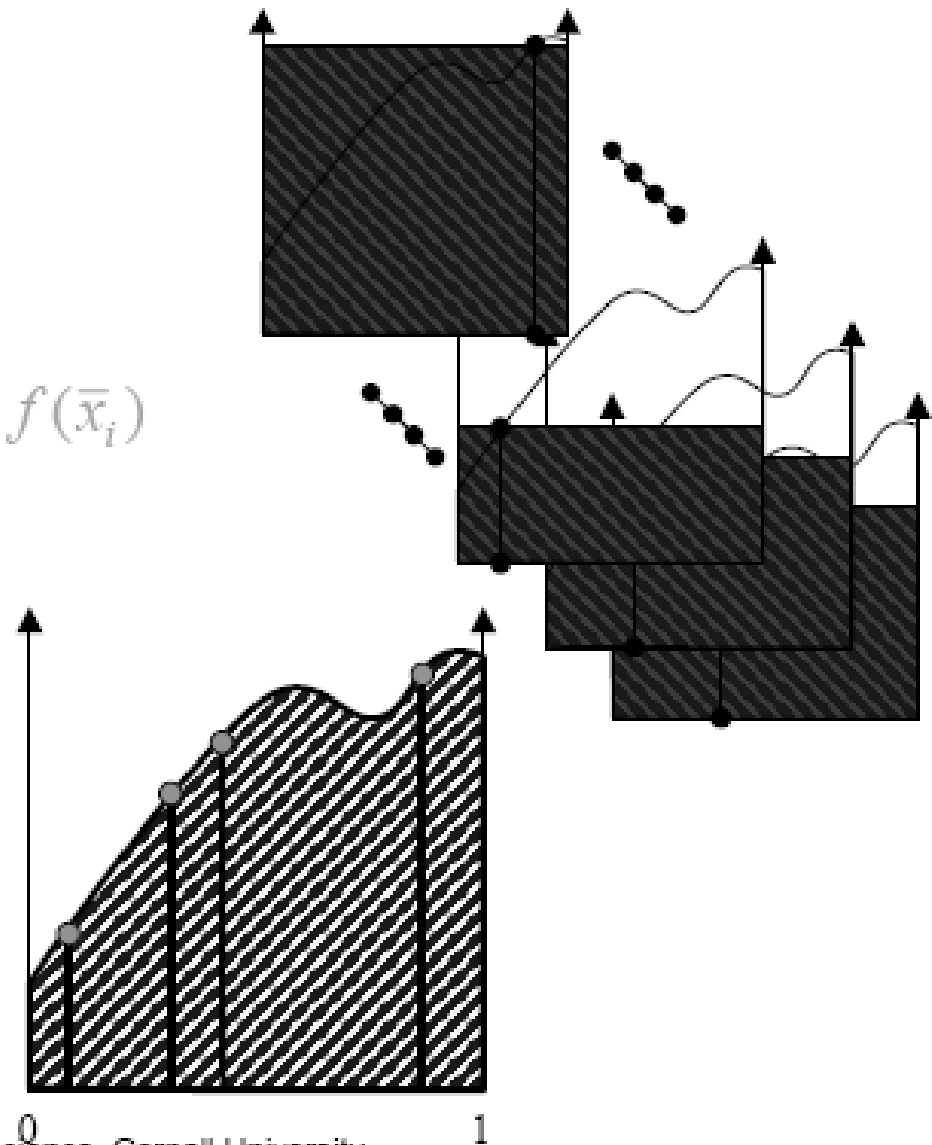
More samples

Secondary estimator

Generate N random samples x_i

Estimator:
$$\langle I \rangle = I_{\text{sec}} = \frac{1}{N} \sum_{i=1}^N f(\bar{x}_i)$$

Variance
$$\sigma_{\text{sec}}^2 = \sigma_{\text{prim}}^2 / N$$



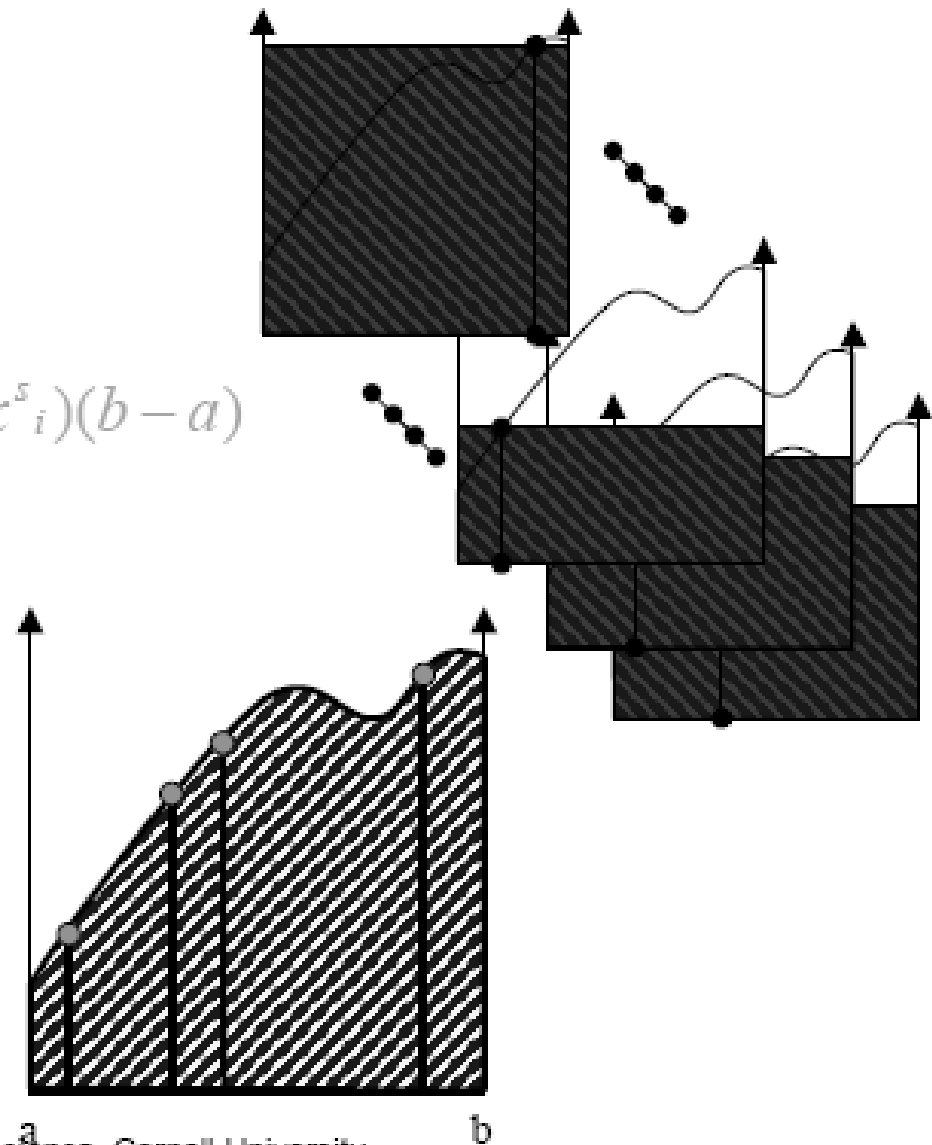
More samples

Secondary estimator

Generate N random samples x_i

$$\text{Estimator: } \langle I \rangle = I_{\text{sec}} = \frac{1}{N} \sum_{i=1}^N f(x_i^s)(b-a)$$

Variance $\sigma_{\text{sec}}^2 = \sigma_{\text{prim}}^2 / N$



Monte Carlo Integration

- Expected value of estimator

$$\begin{aligned} E[\langle I \rangle] &= E\left[\frac{1}{N} \sum_i^N \frac{f(x_i)}{p(x_i)}\right] = \frac{1}{N} \int \left(\sum_i^N \frac{f(x_i)}{p(x_i)}\right) p(x) dx \\ &= \frac{1}{N} \sum_i^N \int \left(\frac{f(x)}{p(x)}\right) p(x) dx \\ &= \frac{N}{N} \int f(x) dx = I \end{aligned}$$

– on ‘average’ get right result: **unbiased**

- Standard deviation σ is a measure of the stochastic error

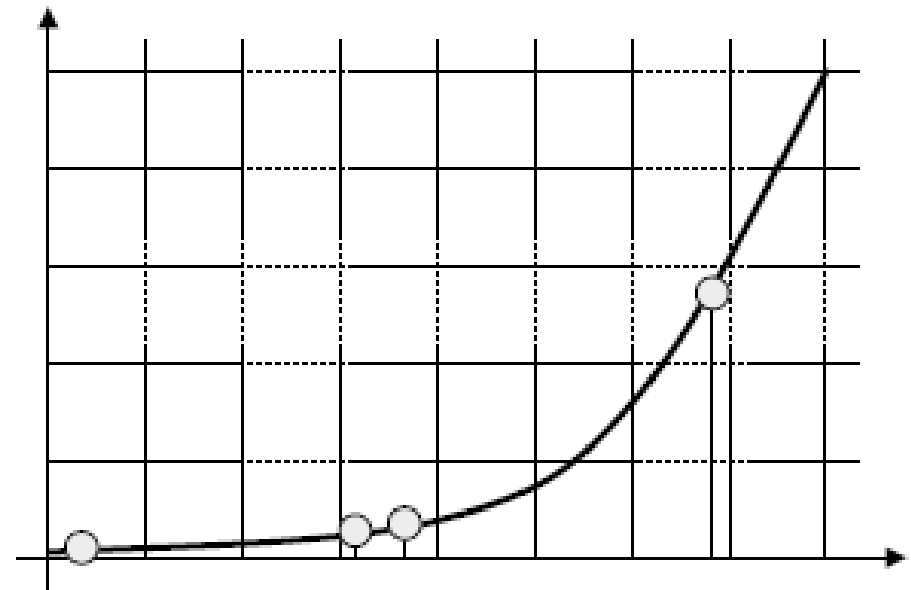
$$\sigma^2 = \frac{1}{N} \int_a^b \left[\frac{f(x)}{p(x)} - I\right]^2 p(x) dx$$

MC Integration - Example

– Integral $I = \int_0^1 5x^4 dx = 1$

– Uniform sampling

– Samples :



$$x_1 = .86 \quad \langle I \rangle = 2.74$$

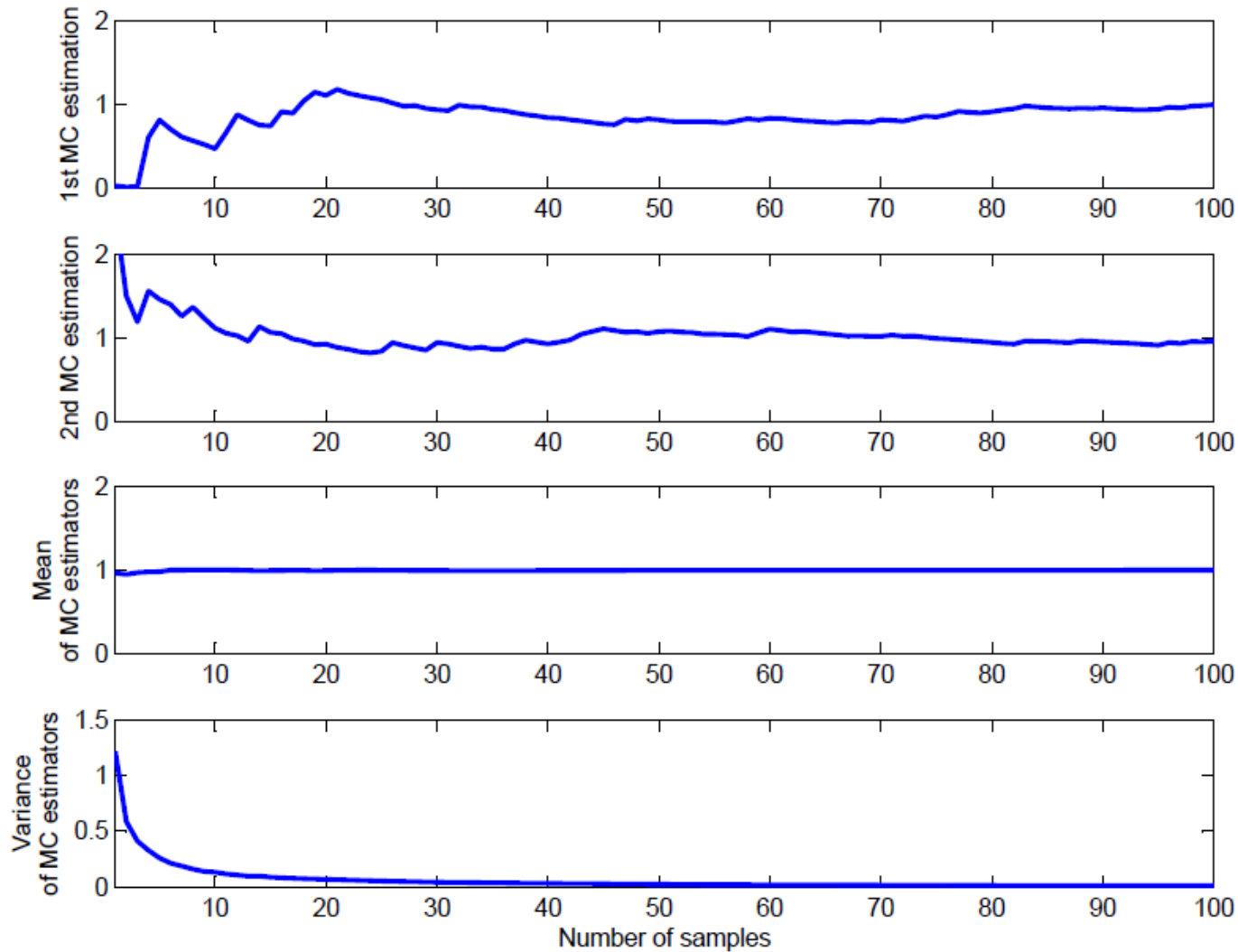
$$x_2 = .41 \quad \langle I \rangle = 1.44$$

$$x_3 = .02 \quad \langle I \rangle = 0.96$$

$$x_4 = .38 \quad \langle I \rangle = 0.75$$

MC Integration - Example

$$I = \int_0^1 4x^3 dx = 1.$$

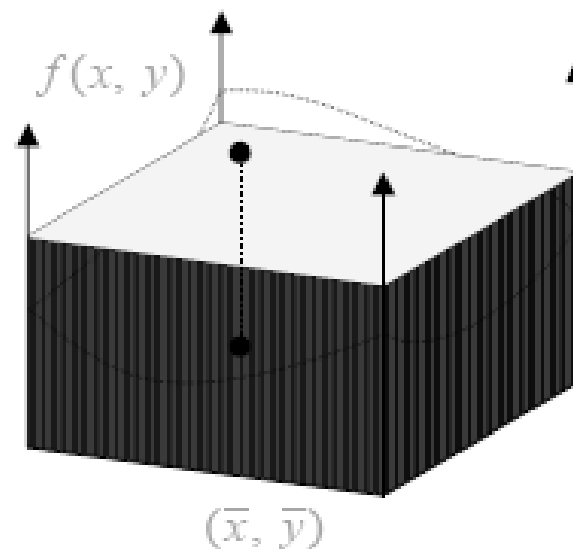


Code:
`mc_int_ex.m`

MC Integration: 2D

- Primary estimator:

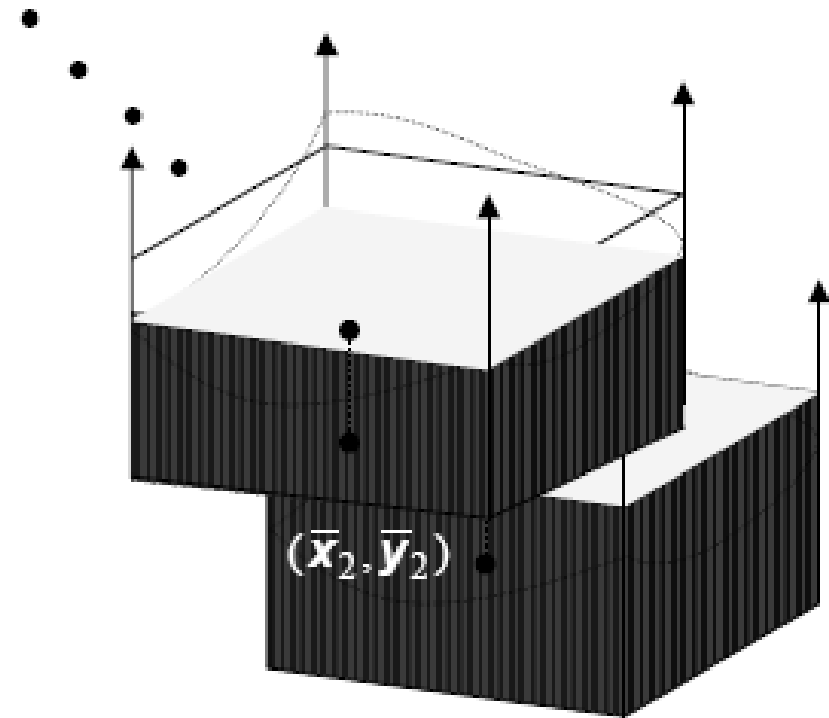
$$\bar{I}_{prim} = \frac{f(\bar{x}, \bar{y})}{p(\bar{x}, \bar{y})}$$



MC Integration: 2D

- Secondary estimator:

$$I_{\text{sec}} = \frac{1}{N} \sum_{i=1}^N \frac{f(\bar{x}_i, \bar{y}_i)}{p(\bar{x}_i, \bar{y}_i)}$$

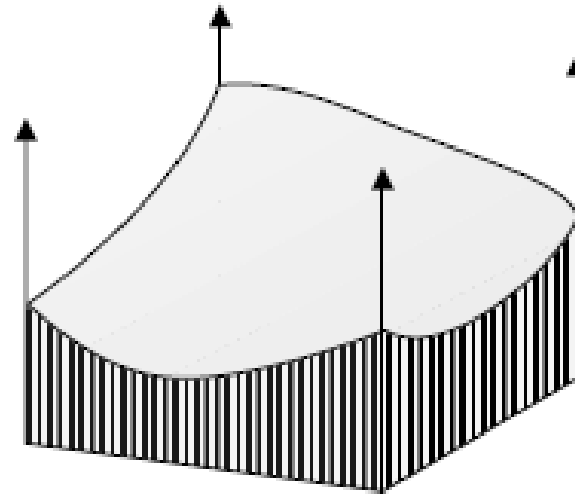


Monte Carlo Integration - 2D

- MC Integration works well for higher dimensions
- Unlike quadrature

$$I = \int_a^b \int_c^d f(x, y) dx dy$$

$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^N \frac{f(x_i, y_i)}{p(x_i, y_i)}$$



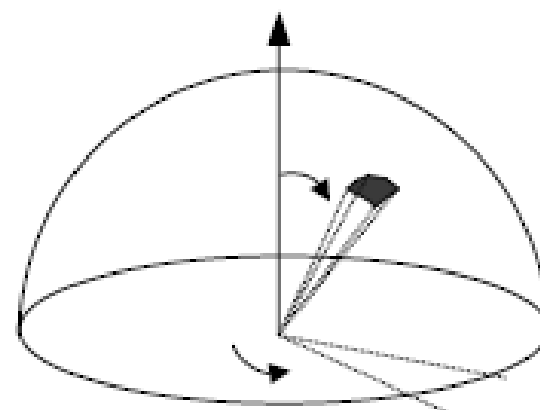
Advantages of MC

- Convergence rate of $O(\frac{1}{\sqrt{N}})$
- Simple
 - Sampling
 - Point evaluation
- General
 - Works for high dimensions
 - Deals with discontinuities, crazy functions, etc.

MC Integration - 2D example

- Integration over hemisphere:

$$\begin{aligned} I &= \int_{\Omega} f(\Theta) d\omega_{\Theta} \\ &= \int_0^{2\pi} \int_0^{\pi/2} f(\varphi, \theta) \sin \theta d\theta d\varphi \end{aligned}$$



$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^N \frac{f(\varphi_i, \theta_i) \sin \theta}{p(\varphi_i, \theta_i)}$$

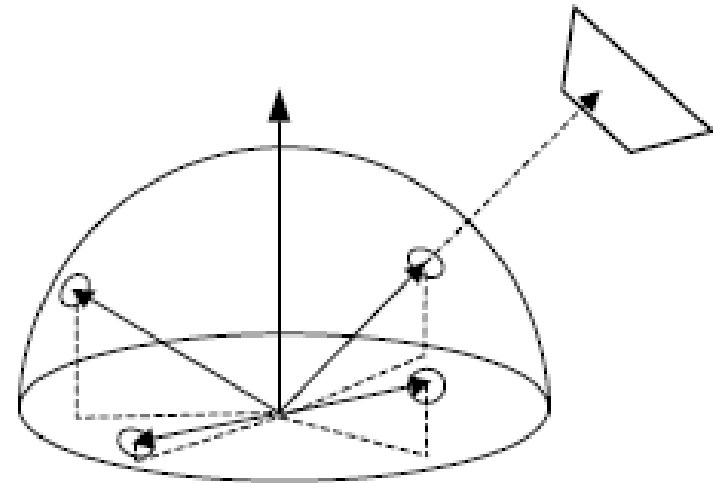
Hemisphere Integration example

Irradiance due to light source:

$$I = \int_{\Omega} L_{source} \cos \theta d\omega_{\ominus}$$
$$= \int_0^{2\pi} \int_0^{\pi/2} L_{source} \cos \theta \sin \theta d\theta d\varphi$$

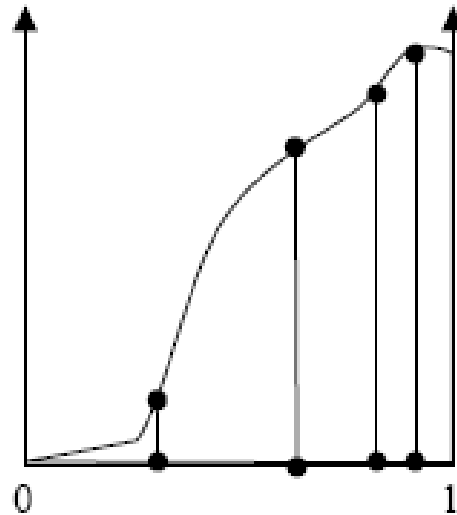
$$p(\omega_i) = \frac{\cos \theta \sin \theta}{\pi}$$

$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^N \frac{L_{source}(\omega_i) \cos \theta \sin \theta}{p(\omega_i)} = \frac{\pi}{N} \sum_{i=1}^N L_{source}(\omega_i)$$



Importance Sampling

- Take more samples in important regions, where the function is large



From kavita's slides

MC integration - Non-Uniform

- Some parts of the integration domain have higher importance
- Generate samples according to density function $p(x)$

$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^N \frac{f(x_i)}{p(x_i)}$$

- Estimator?

- What is optimal $p(x)$? $p(x) \approx f(x) / \int f(x) dx$

MC integration - Non-Uniform

- Generate samples according to density function $p(x)$

$$p(x) \approx f(x) / \int f(x) dx$$

- Why? $I_{estimator} = \frac{1}{N} \sum \frac{f(x)}{p(x)} = \frac{1}{N} \sum \frac{f(x)}{f(x)/I} = \frac{1}{N} \sum I = I$

$$\sigma^2 = \frac{1}{N} \int_a^b \left[\frac{f(x)}{p(x)} - I \right]^2 p(x) dx$$

- But.....

$$= \frac{1}{N} \int_a^b \left[\frac{f(x)}{f(x)/I} - I \right]^2 p(x) dx = 0$$

Example

- Function: $I = \int_0^4 x dx = 8$ $f(x) = x$
 $\sigma^2 = \frac{1}{N} \int_a^b \left[\frac{f(x)}{p(x)} - I \right]^2 p(x) dx$

$$p(x) = \frac{x}{8}, \sigma^2 = 0 \quad I_{\text{estimator}} = I = 8$$

$$p(x) = \frac{1}{4}, \sigma^2 = \frac{1}{N} \int_0^4 \left[\frac{x}{1/4} - 8 \right]^2 \frac{1}{4} dx = 21.3 / N$$

$$p(x) = \frac{x+2}{16}, \sigma^2 = \frac{1}{N} \int_0^4 \left[\frac{x}{(x+2)/16} - 8 \right]^2 \frac{x+2}{16} dx = 6.3 / N$$

Importance Sampling

- Generate samples from density function $p(x)$

$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^N \frac{f(x_i)}{p(x_i)}$$

- Optimal $p(x)$? $p(x) \approx f(x) / \int f(x) dx$
- General principle:
 - Closer shape of $p(x)$ is to shape of $f(x)$, lower the variance
- Variance can *increase* if $p(x)$ is chosen badly

Sampling according to pdf

- Inverse cumulative distribution function
- Rejection sampling

Inverse Cumulative Distribution Function – Discrete Case

- Consider discrete events x_i – with probability p_i
- Select x_i if:

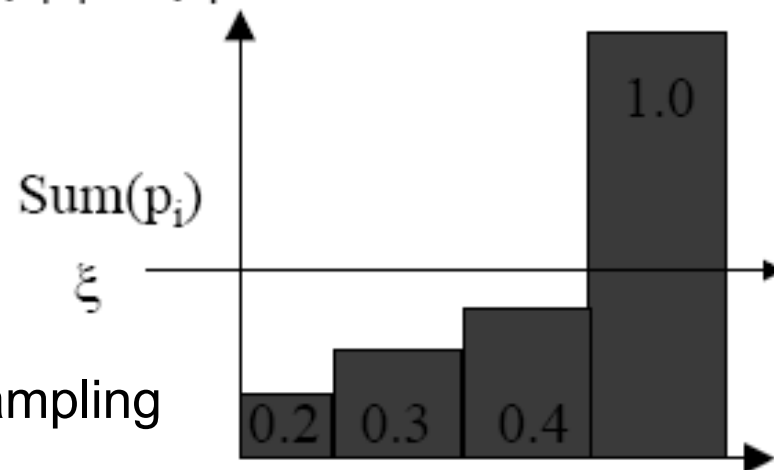
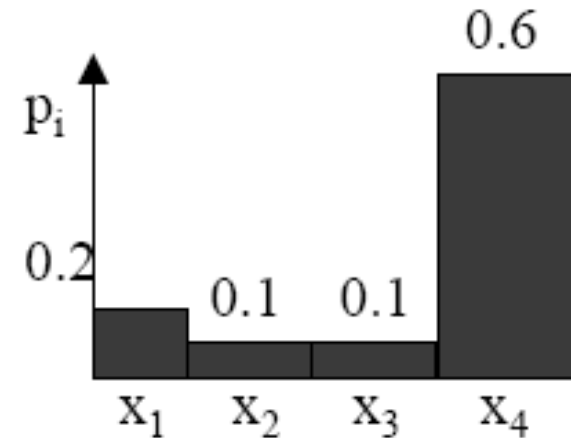
$$p_1 + \dots + p_{i-1} < \xi < p_1 + \dots + p_{i-1} + p_i$$

$$\sum_{j=1}^{i-1} p_j < \xi < \sum_{j=1}^i p_j$$

$$P(x_i) = P(\xi \in [\sum_{j=1}^{i-1} p_j, \sum_{j=1}^i p_j])$$

$P(a < \xi < b) = (b - a)$, given uniform sampling

$$P(x_i) = P(\xi \in [\sum_{j=1}^{i-1} p_j, \sum_{j=1}^i p_j]) = \sum_{j=1}^i p_j - \sum_{j=1}^{i-1} p_j = p_i$$



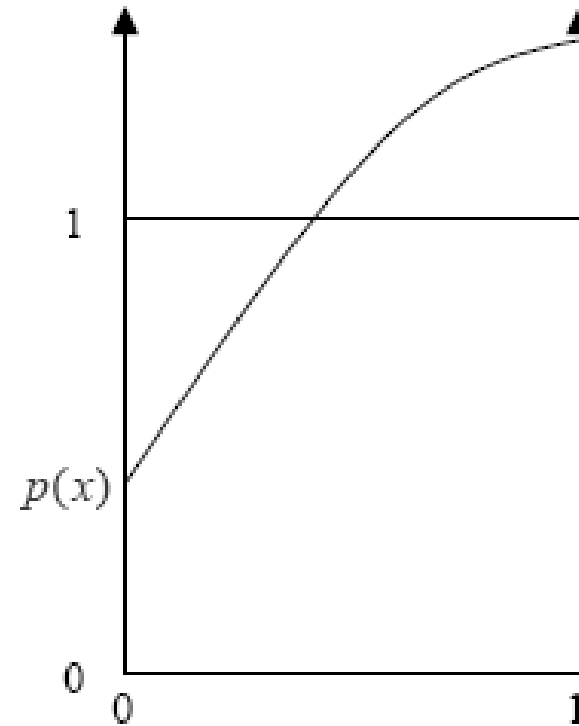
Continuous Random Variable

- **Algorithm**

- Pick u uniformly from $[0, 1)$
- Output $y = P^{-1}(u)$, where $P(y) = \int_{-\infty}^y p(x)dx$

Non-Uniform Samples

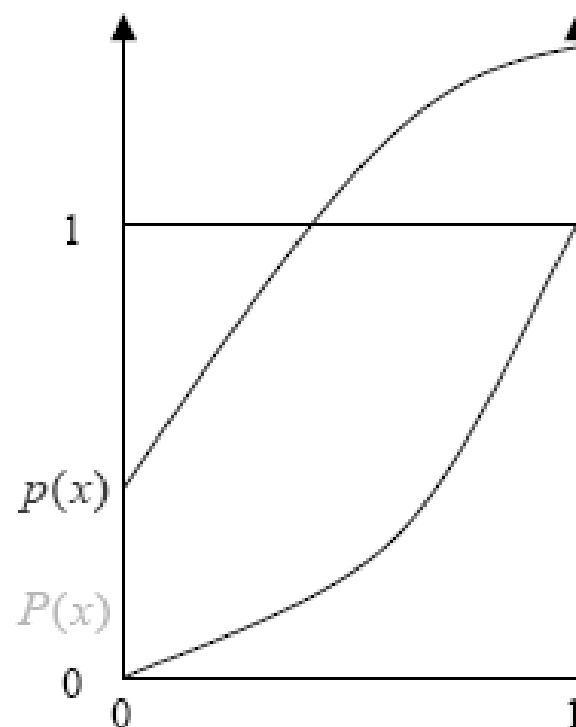
- 1) Choose a normalized probability density function $p(x)$



Non-Uniform Samples

- 1) Choose a normalized probability density function $p(x)$
- 2) Integrate to get a cumulative probability distribution function $P(x)$:

$$P(x) = \int_0^x p(t) dt$$



Note this is similar to computing $\sum_{j=1}^i p_j$

Non-Uniform Samples

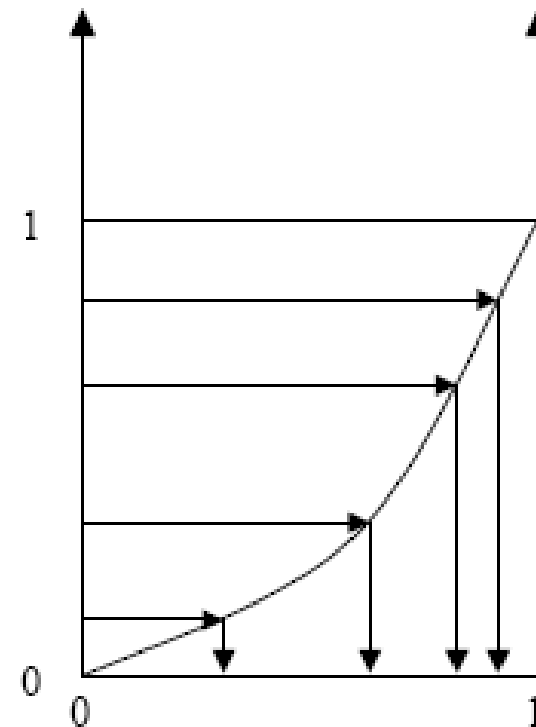
- 1) Choose a normalized probability density function $p(x)$
- 2) Integrate to get a probability distribution function $P(x)$:

$$P(x) = \int_0^x p(t) dt$$

- 3) Invert P :

$$x = P^{-1}(\xi)$$

Note this is similar to going from y axis to x in discrete case!



Cosine distribution

$$f = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \cos \theta \sin \theta d\theta d\phi$$

$$p(\theta, \phi) = \frac{\cos \theta \sin \theta}{\pi}$$

$$CDF(\theta, \phi) = \int_0^\theta \int_0^\phi \frac{\cos \theta \sin \theta}{\pi} d\theta d\phi = (1 - \cos^2 \theta) \frac{\phi}{2\pi}$$

$$F(\theta) = 1 - \cos^2 \theta$$

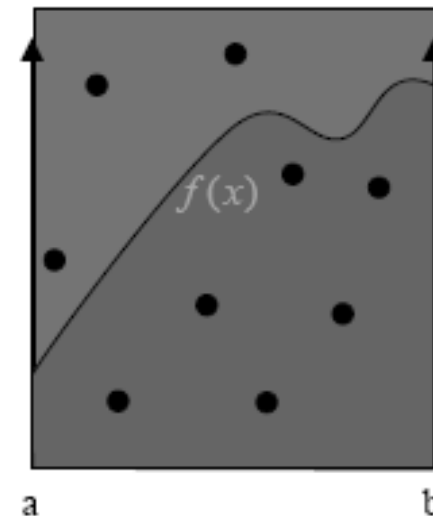
$$F(\phi) = \frac{\phi}{2\pi}$$

$$\phi_i = 2\pi\xi_1 \quad \theta_i = \cos^{-1} \sqrt{\xi_2}$$

Rejection Method

- Often not possible to compute the inverse of cdf
- Pick ξ_1, ξ_2

$$I = \int_a^b f(x) dx$$



From kavita's slides

- If $\xi_2 < f(\xi_1)$, select ξ_1
- Is this efficient? What determines efficiency? $A(f)/A(\text{rectangle})$

Class Objectives were:

- **Sampling approach for solving the rendering equation**
 - Monte Carlo integration
 - Estimator and its variance
 - Sampling according to the pdf

Homework

- **Go over the next lecture slides before the class**
- **Watch 2 SIGGRAPH videos and submit your summaries every Tue. class**
 - **Just one paragraph for each summary**

Example:

Title: XXX XXXX XXXX

Abstract: this video is about accelerating the performance of ray tracing. To achieve its goal, they design a new technique for reordering rays, since by doing so, they can improve the ray coherence and thus improve the overall performance.

Any Questions?

- **Come up with one question on what we have discussed in the class and submit at the end of the class**
 - 1 for already answered questions
 - 2 for typical questions
 - 3 for questions with thoughts
- **Submit questions at least four times before the mid-term exam**
 - Multiple questions for the class is counted as only a single time

Next Time

- Monte Carlo ray tracing