CS580: Student Presentation
mipNeRF & mipNeRF360

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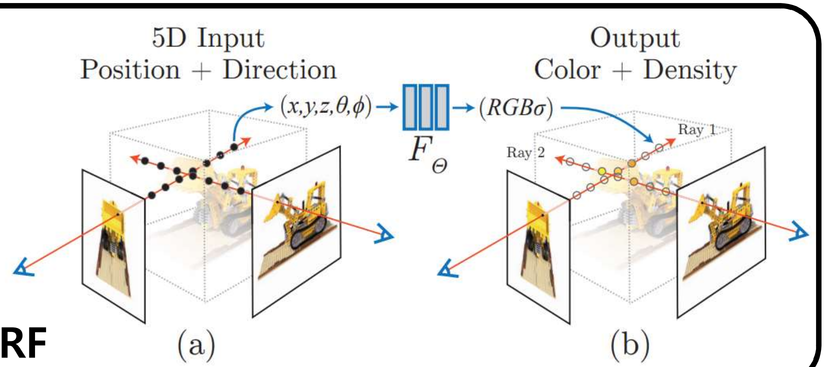
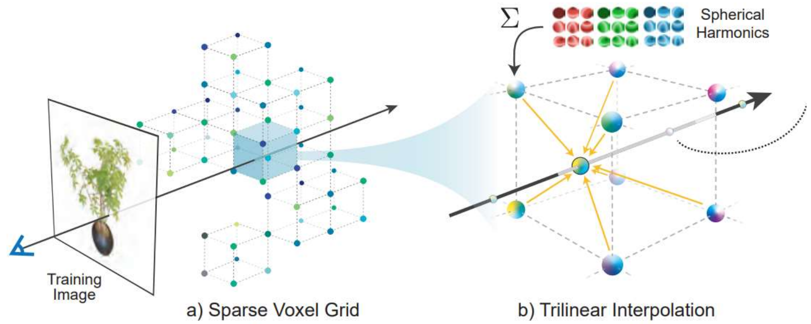
(최동영)

Recap. Previous Presentation



NeRF

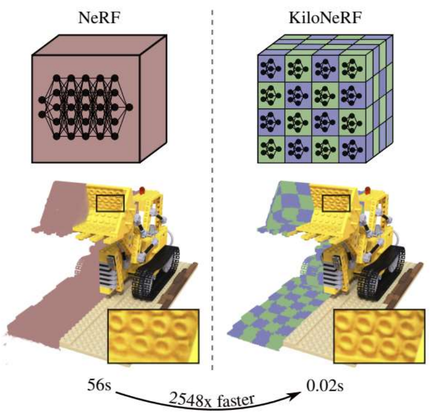
Acceleration by **sparse voxel grid**



Plenoxels

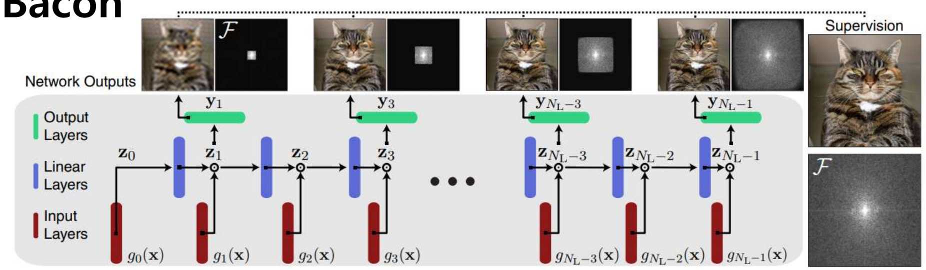
Acceleration by **thousand of small MLPs**

Can be applied to **neural radiance field**



KiloNeRF

Bacon



CS580: Student Presentation
mipNeRF

ICCV 2021 Oral, Best Paper Honorable Mention
Jonathan T. Barron et al.

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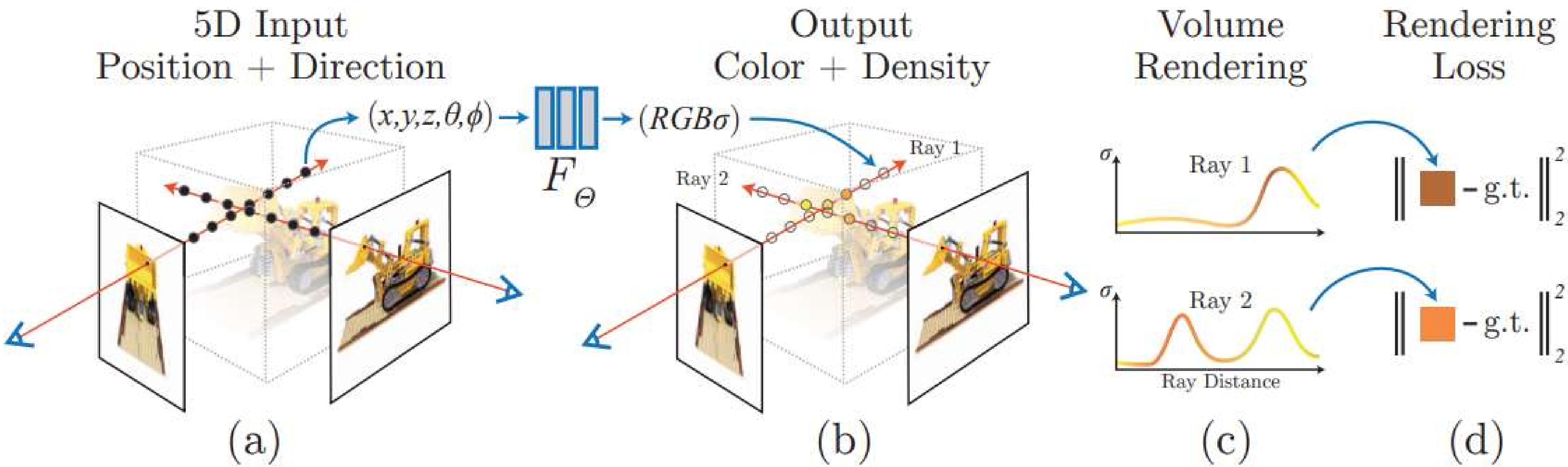
Recap. NeRF

- **Render 3D object using 2D Images with camera matrix by optimizing radiance field equation via neural network**



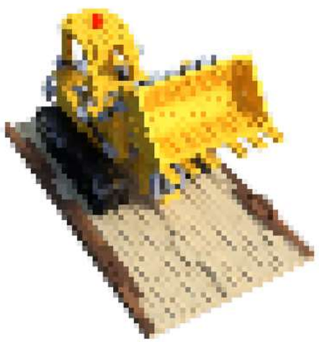
Recap. NeRF (conti.)

- NeRF can even generate unobserved view of trained object

Result of NeRF

Aliasing in multi-resolution images

- Aliasing occurs when NeRF learns **low resolution(1/2, ¼, …) images**



NeRF

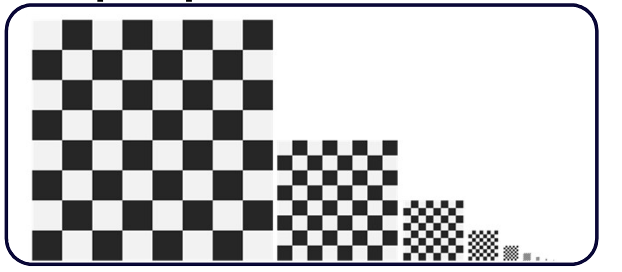
mipNeRF

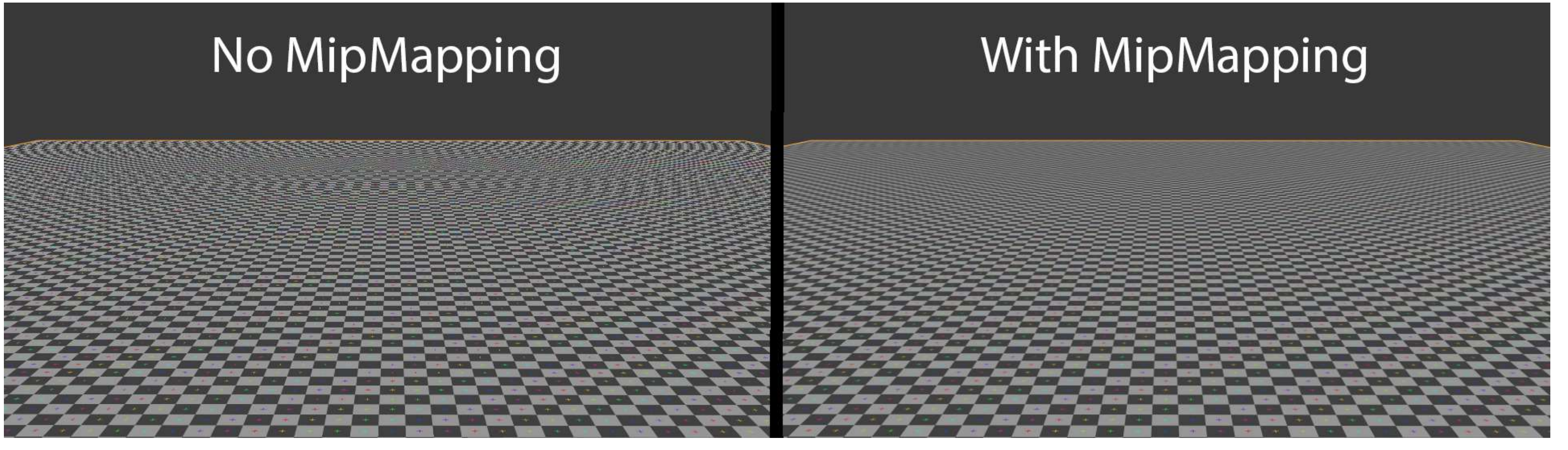
Ground Truth **NeRF**

Mipmap

- Aliasing effects can be reduced by using pre-computed gaussian filtered images **(prefiltering)**

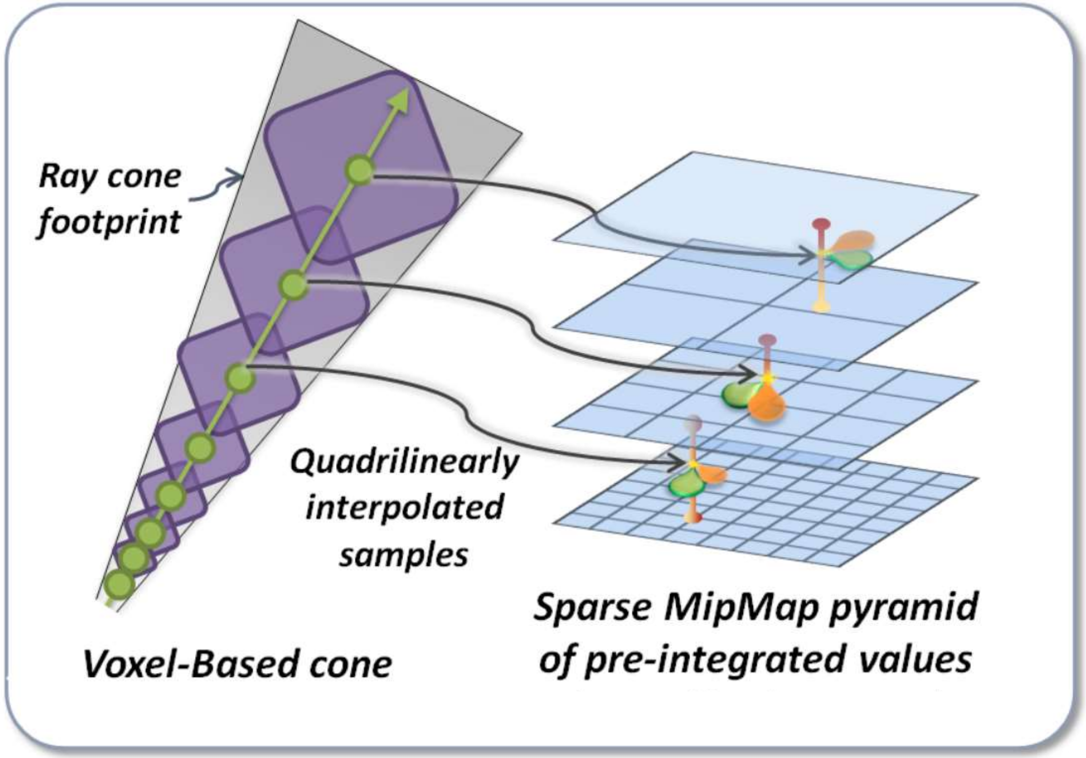
mipmap





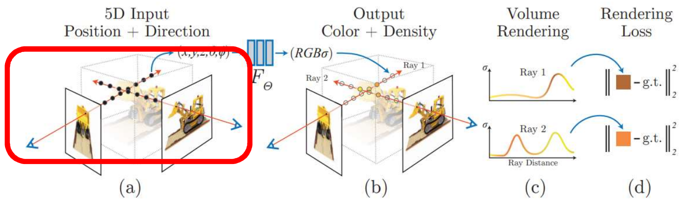
Cone tracing

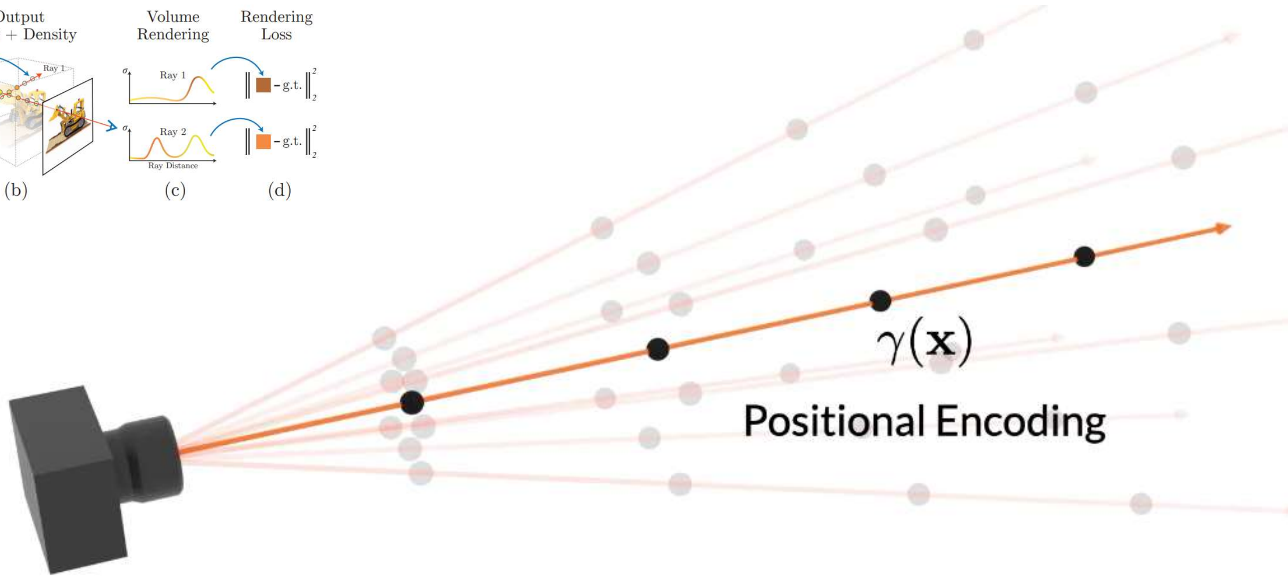
- mipNeRF uses **cone tracing** that acts like mipmap in rendering



Ray casting in NeRF

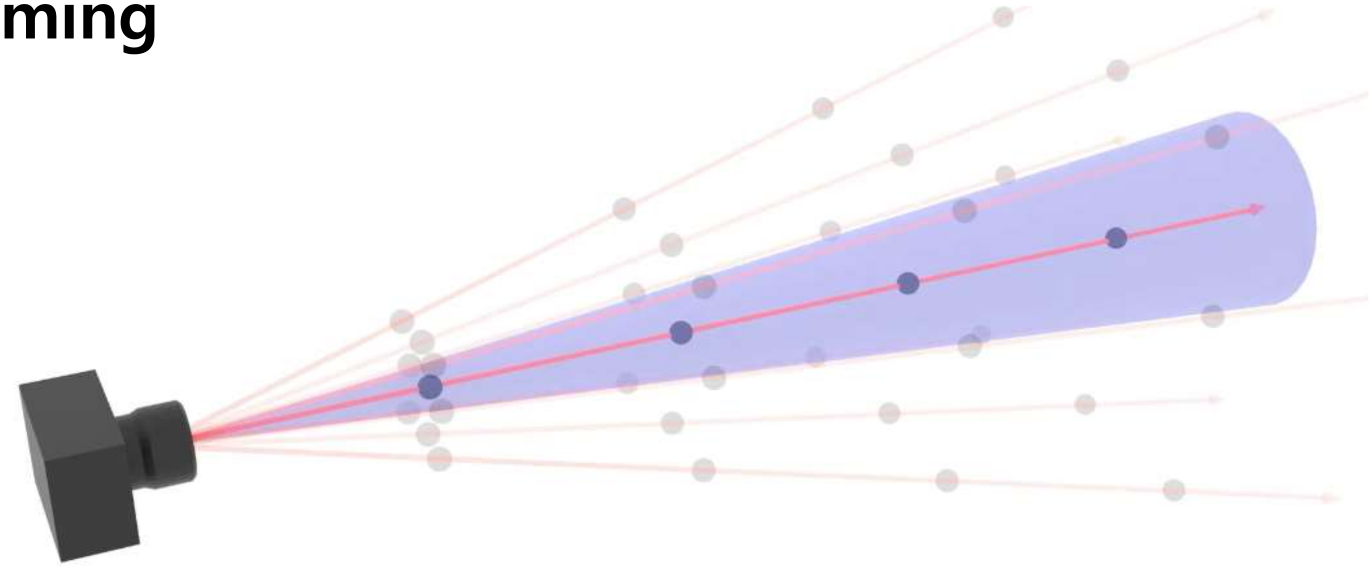
- NeRF casts a **ray** for sampling the points





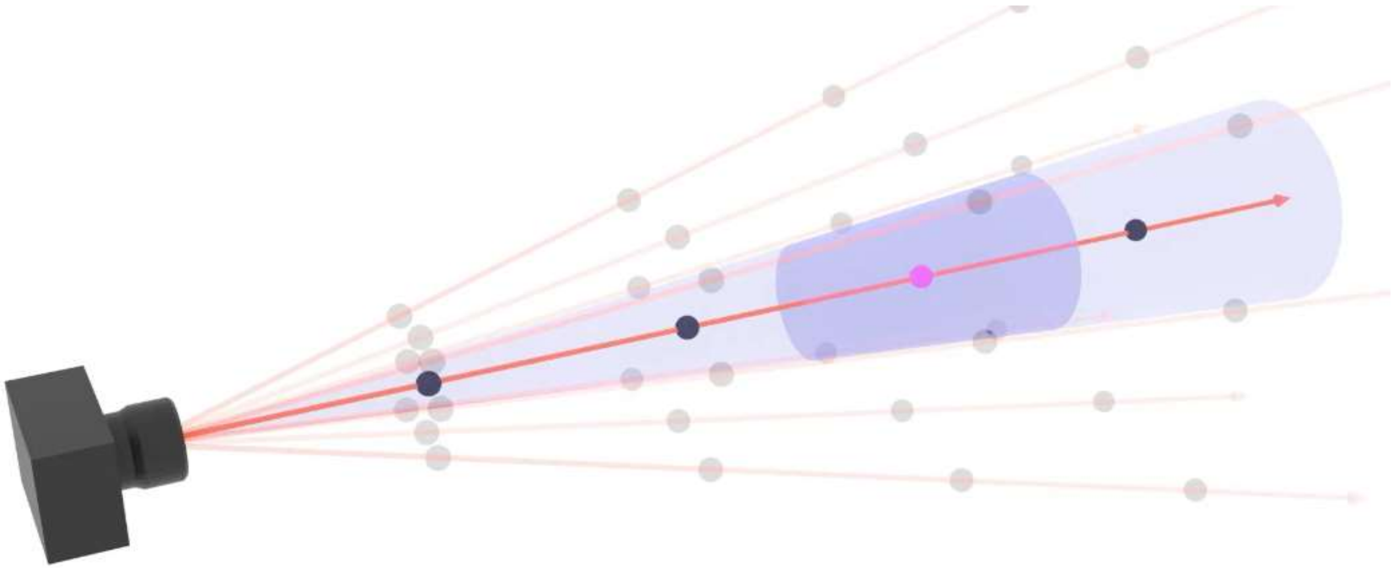
Cone casting in mipNeRF

- Instead, mipNeRF casts a **cone** to samples
- But sampling all points in the cones is **extremely time consuming**



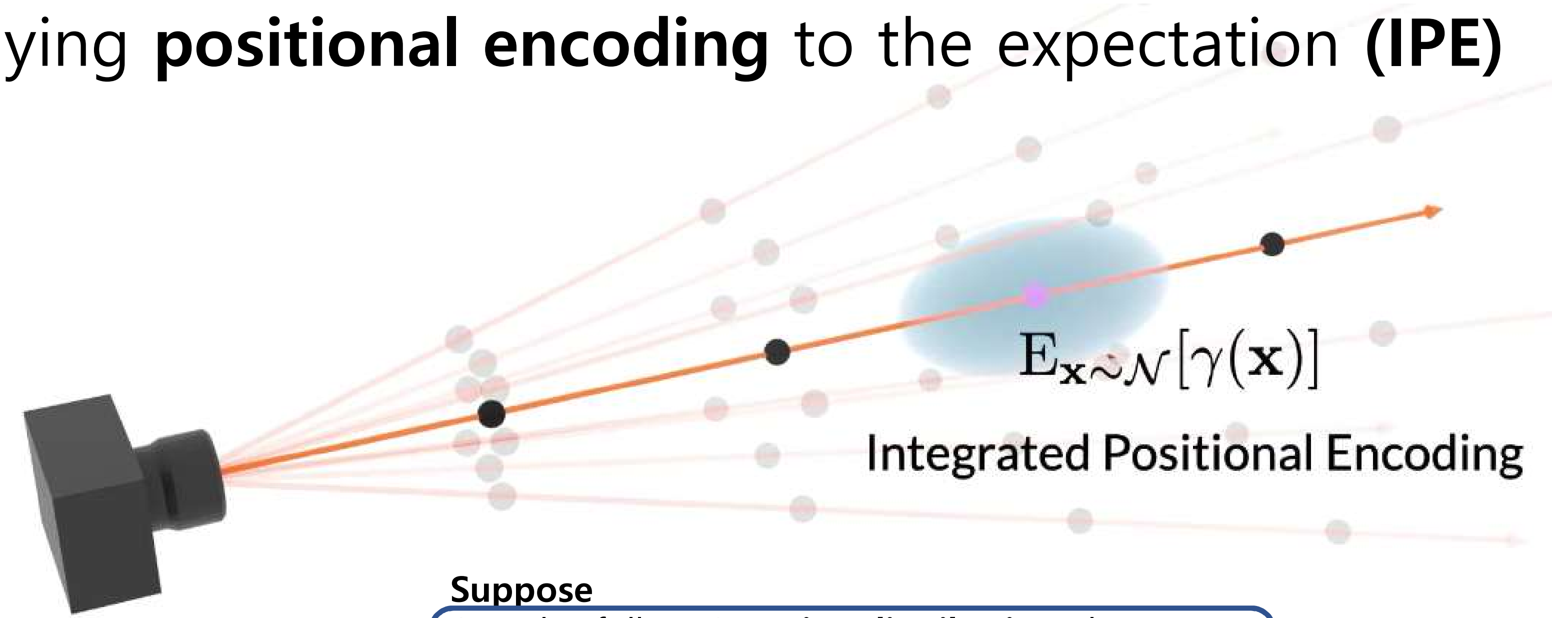
Cone casting in mipNeRF (cont.)

- Divide cone as **multiple conical frustum**



Cone casting in mipNeRF (cont.)

- Use the **expectation** of each conical frustum as samples!
- Suppose all samples follow the **Gaussian distribution**
- Applying **positional encoding** to the expectation **(IPE)**



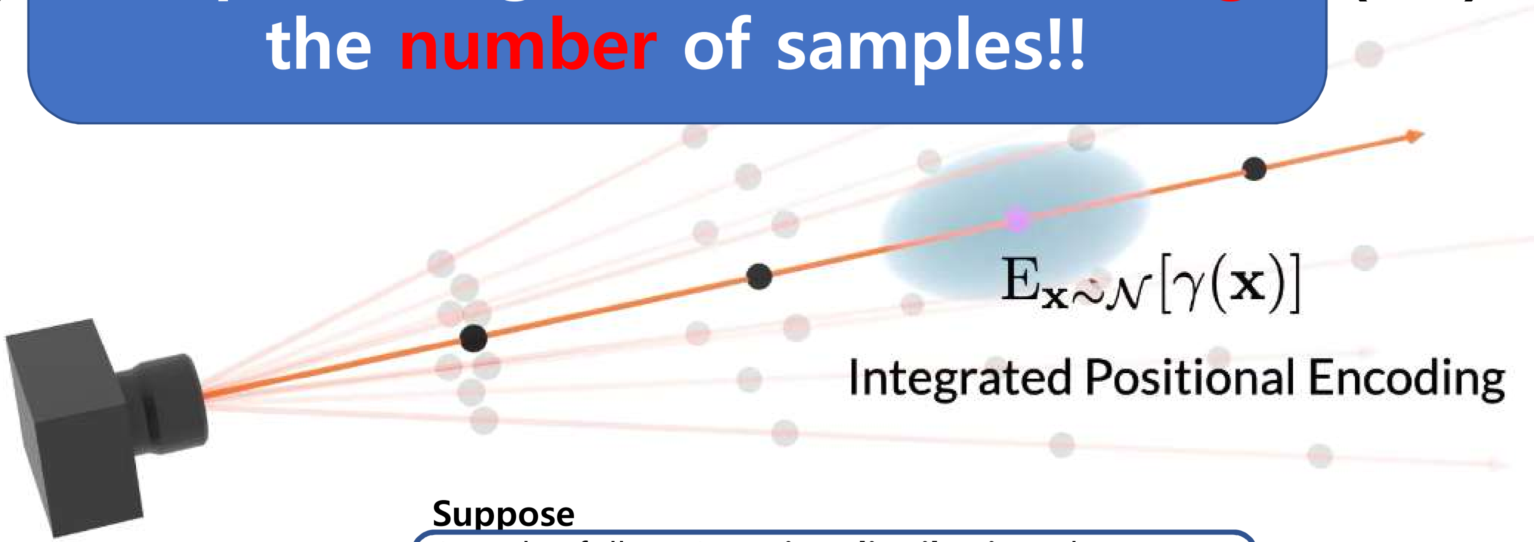
Suppose

Samples follow **Gaussian distribution** along a ray direction and it vertical one (multivariate Gaussian distribution)

Cone casting in mipNeRF (cont.)

- Use the **expectation** of each conical frustum as samples!
- Suppo … **ution**
- Applyi … **(IPE)**

Achieve higher performance with simple changes without increasing the number of samples!!



Suppose

Samples follow **Gaussian distribution** along a ray direction and it vertical one (multivariate Gaussian distribution)

Integrated Positional Encoding (IPE)

Positional Encoding

$$P = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 0 & & 2^{L-1} & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & \cdots & 0 & 2^{L-1} & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & & 0 & 0 & 2^{L-1} \end{bmatrix}^T, \quad \gamma(\mathrm{x}) = \begin{bmatrix} \sin(\mathrm{Px}) \\ \cos(\mathrm{Px}) \end{bmatrix}$$

Conical frustum equation

Expectation

Positional encoding

$$E_{\mathrm{x} \sim N(\mu, \sigma^2)}[\gamma(\mathrm{x})]$$

Normal distribution

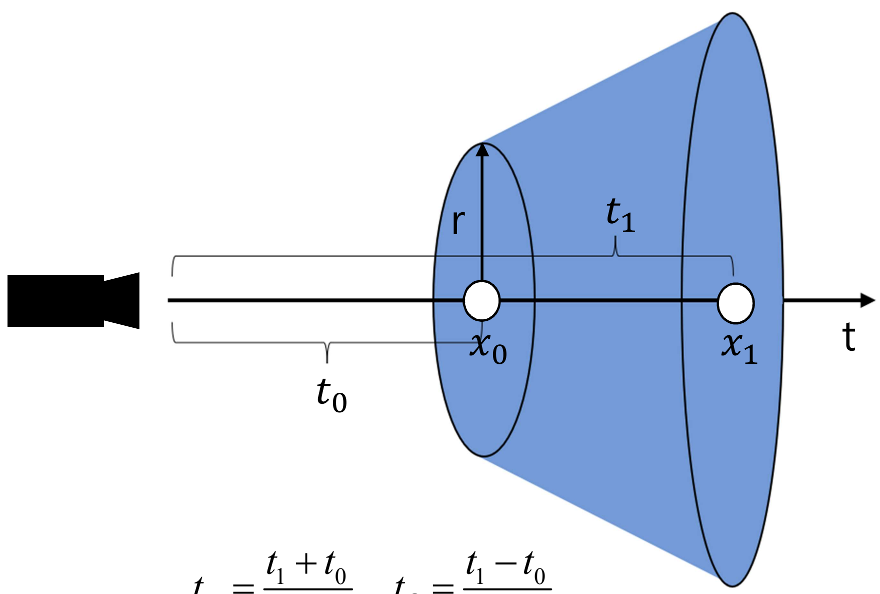
~

$$F(x, o, d, \dot{r}, t_0, t_1) = \left\{ \left(t_0 < \frac{d^T(x-o)}{\|d\|_2^2} < t_1 \right) \wedge \left(\frac{d^T(x-o)}{\|d\|_2 \|x-o\|_2} > \frac{1}{\sqrt{1 + (\dot{r} / \|d\|_2)^2}} \right) \right\}$$

x are the member of conical frustum

Integrated Positional Encoding (IPE) (cont.)

- Mean and **Var** is determined by the **distance** between samples!



$$t_\mu = \frac{t_1 + t_0}{2}, \quad t_\delta = \frac{t_1 - t_0}{2}$$

Mean, Covariance matrix of conical frustum

$$\mu_t = t_\mu + \frac{2 t_\mu t_\delta^2}{3t_\mu^2 + t_\delta^2}, \quad \sigma_t^2 = \frac{t_\delta^2}{3} - \frac{4t_\delta^2(12t_\mu^2 - t_\delta^2)}{15(t_\mu^2 + t_\delta^2)^2},$$

$$\sigma_r^2 = \dot{r}^2\left(\frac{t_\mu^2}{4} + \frac{5t_\delta^2}{12} - \frac{4t_\delta^4}{15(3t_\mu^2 + t_\delta^2)}\right)$$

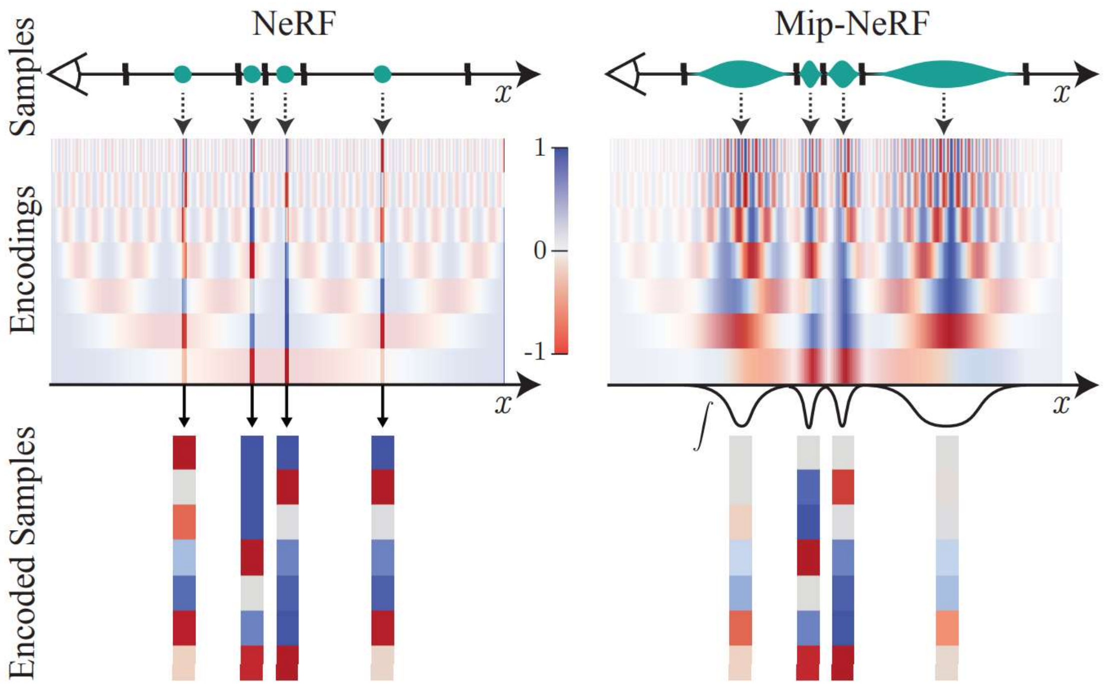
cone dir

radius dir

$$\mu = o + \mu_t d, \quad \Sigma = \sigma_t(dd^{\mathrm{T}}) + \sigma_r\left(I - \frac{dd^{\mathrm{T}}}{\|d\|_2^2}\right)$$

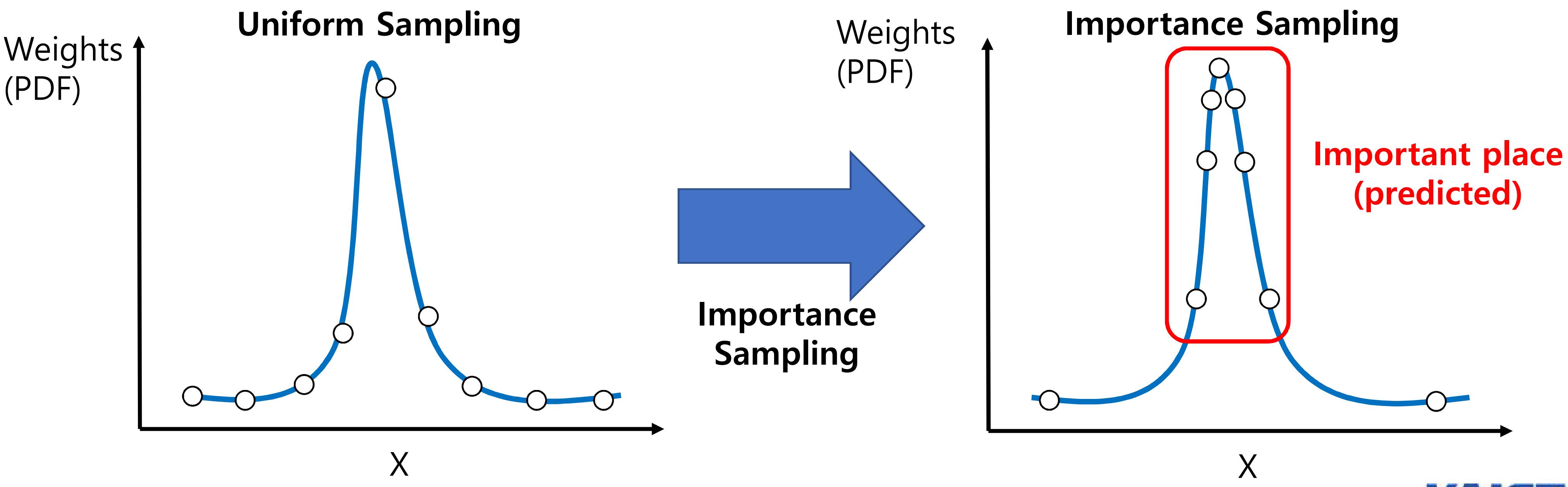
Result of IPE

- Get the samples including **variance information** implicitly



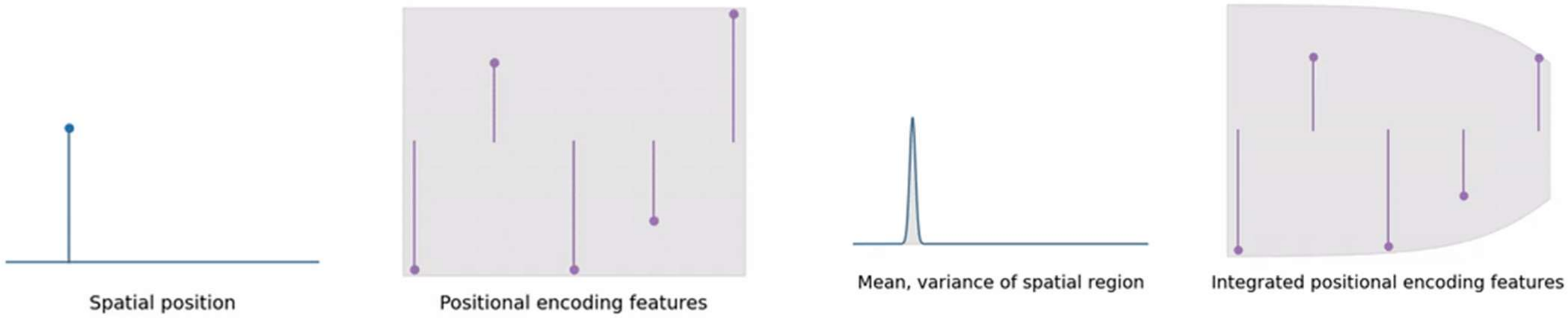
Importance Sampling

- Increase the number of samples predicted to be important
- **The important samples distance would be really short!**



Result of IPE (conti.)

- The distance between samples is **large**
  -> **cut off** the high freq. (**details**)
- The distance between samples is **short**
  -> **maintain** the high freq. (**details**)



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Result of IPE (conti.)

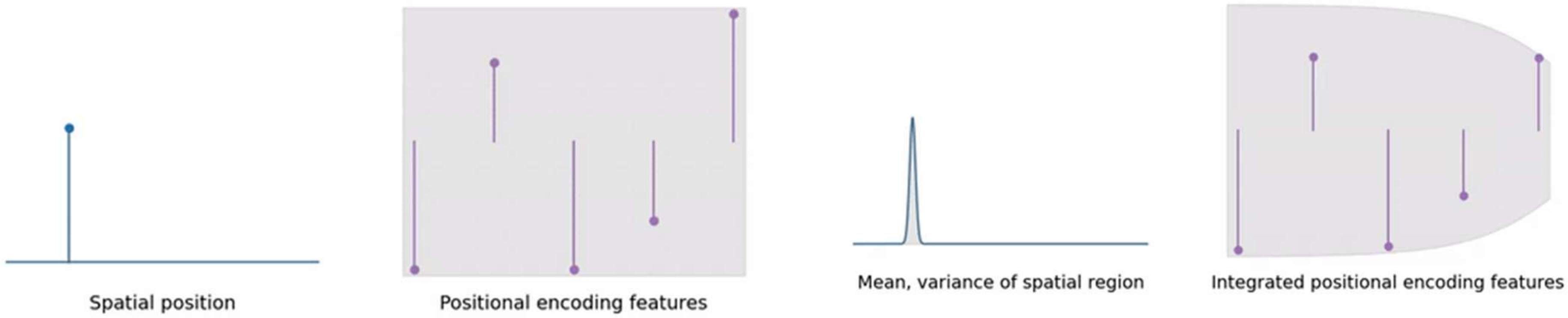
- The distance between samples is **large**

  -> **cu**

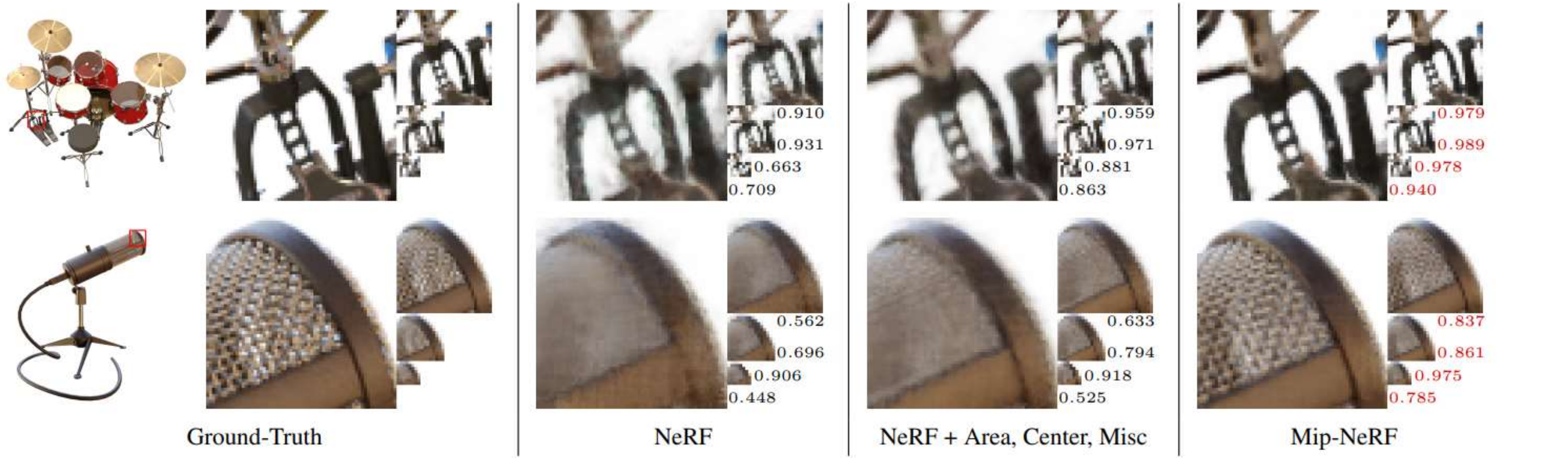
- The dista

  -> **m**

Keep the details of important samples and omit the details of unimportant samples autonomously!



Result



	PSNR ↑				SSIM ↑				LPIPS ↓						
	Full Res.	1/2 Res.	1/4 Res.	1/8 Res.	Full Res.	1/2 Res.	1/4 Res.	1/8 Res.	Full Res.	1/2 Res.	1/4 Res.	1/8 Res	Avg. ↓	Time (hours)	# Params
NeRF (Jax Impl.) [11, 30]	31.196	30.647	26.252	22.533	0.9498	0.9560	0.9299	0.8709	0.0546	0.0342	0.0428	0.0750	0.0288	3.05 ± 0.04	1,191K
NeRF + Area Loss	27.224	29.578	29.445	25.039	0.9113	0.9394	0.9524	0.9176	0.1041	0.0677	0.0406	0.0469	0.0305	3.03 ± 0.03	1,191K
NeRF + Area, Centered Pixels	29.893	32.118	33.399	29.463	0.9376	0.9590	0.9728	0.9620	0.0747	0.0405	0.0245	0.0398	0.0191	3.02 ± 0.05	1,191K
NeRF + Area, Center, Misc.	29.900	32.127	33.404	29.470	0.9378	0.9592	0.9730	0.9622	0.0743	0.0402	0.0243	0.0394	0.0190	2.94 ± 0.02	1,191K
Mip-NeRF	32.629	34.336	35.471	35.602	0.9579	0.9703	0.9786	0.9833	0.0469	0.0260	0.0168	0.0120	0.0114	2.84 ± 0.01	612K
Mip-NeRF w/o Misc.	32.610	34.333	35.497	35.638	0.9577	0.9703	0.9787	0.9834	0.0470	0.0259	0.0167	0.0120	0.0114	2.82 ± 0.03	612K
Mip-NeRF w/o Single MLP	32.401	34.131	35.462	35.967	0.9566	0.9693	0.9780	0.9834	0.0479	0.0268	0.0169	0.0116	0.0115	3.40 ± 0.01	1,191K
Mip-NeRF w/o Area Loss	33.059	34.280	33.866	30.714	0.9605	0.9704	0.9747	0.9679	0.0427	0.0256	0.0213	0.0308	0.0139	2.82 ± 0.01	612K
Mip-NeRF w/o IPE	29.876	32.160	33.679	29.647	0.9384	0.9602	0.9742	0.9633	0.0742	0.0393	0.0226	0.0378	0.0186	2.79 ± 0.01	612K

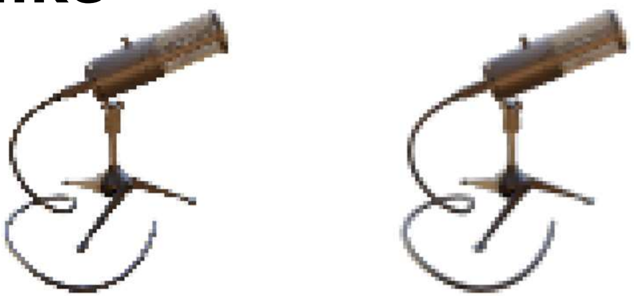
Result

- mipNeRF shows better performance in reducing aliasing!

Lego

Ship

Mike



Chair

Summary of mipNeRF

- Use **cone tracing** instead of ray for **mipmapping**
- Compute **expectations** of samples without sampling whole data
- Propose **Integrated Positional Encoding (IPE)** which autonomously adjusts the details of samples according to the importance

CS580: Student Presentation
mipNeRF 360

CVPR 2022 Oral
Jonathan T. Barron et al.

Obstacles in NeRF at 360˚ Scene

- **Parameterization**

  Samples of 360˚scene would be located from 0 to infinity $(0, \infty)$ (Unbounded scene)

  ← Scene and ray parameterization

- **Efficiency**

  Huge model would be required but need long training time

  ← Coarse-to-Fine Online Distillation

- **Ambiguity**

  Since the sample range is too broad, predicting object geometry is challenging

  ← Regularization for Interval-Based Models

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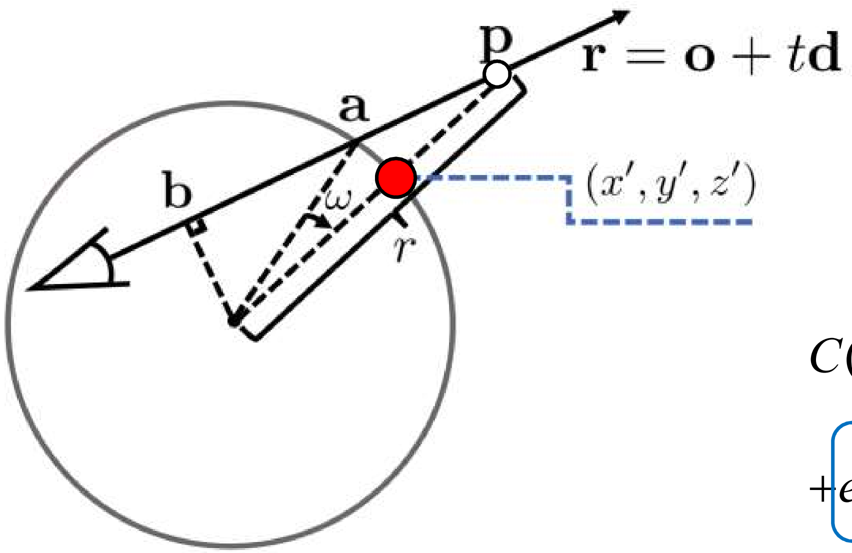
NeRF++: Unbounded scene (2020, arXiv)

Parameterization

- If sample distance > 1:
  use the color&density of samples projected to the sphere



Use the color and density of red points when computing white samples

$$C(r) = \underbrace{\int_{t=0}^{t'} \sigma(o+td)\cdot c(o+td,d)\cdot e^{-\int_{s=0}^{t}\sigma(o+sd)ds}\,dt}_{\text{Inside}}$$

$$+\underbrace{e^{-\int_{s=0}^{t'}\sigma(o+sd)ds}}_{\text{Inside}}\underbrace{\int_{t=t'}^{\infty} \sigma_{out}(o+td)\cdot c_{out}(o+td,d)\cdot e^{-\int_{s=t'}^{t}\sigma_{out}(o+sd)ds}\,dt.}_{\text{Outside}}$$

Inside

Outside

DONeRF: Unbounded scene (2021, Eurographics)

Parameterization

- Sampling **logarithmically** and then **warping** the samples

Uniform sampling

$$\mathbf{x}(d_i) = \mathbf{o} + d_i \cdot \mathbf{r}$$

$$d_i = \left(d_{min} + i \cdot \frac{(d_{max} - d_{min})}{N} \right), \; i = [0, 1, 2, \cdots, N],$$

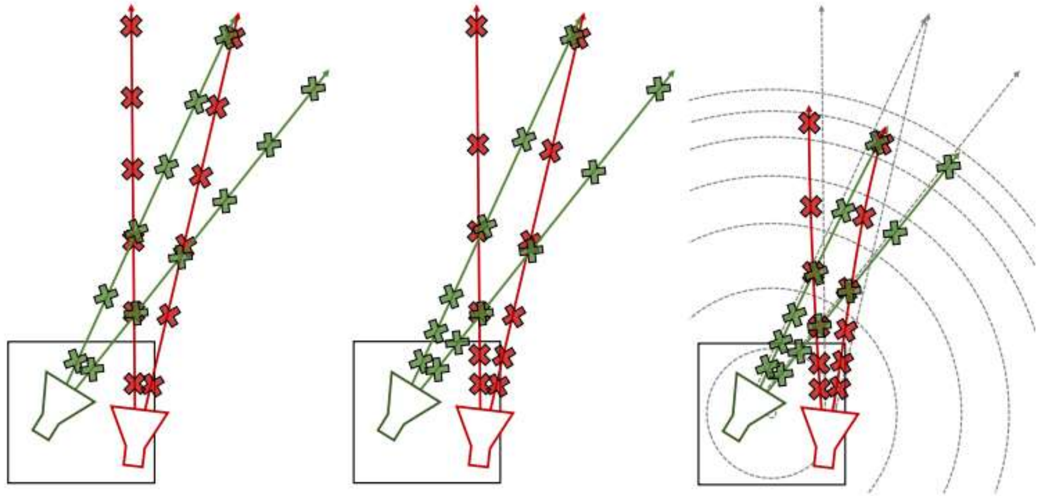
Logarithmic sampling

$$\tilde{d}_i = d_{min} + \frac{\log(d_i - d_{min} + 1)}{\log(d_{max} - d_{min} + 1)} \cdot (d_{max} - d_{min}).$$

Log + Warping

$$f = PE.\left(x(\tilde{d}_i - c) \cdot W(x(\tilde{d}_i - c) \right)$$

$$W(x) = \frac{1}{\sqrt{|x| \cdot d_{\max}}}$$



(a) *uniform* (b) *logarithmic* (c) *log+warp*

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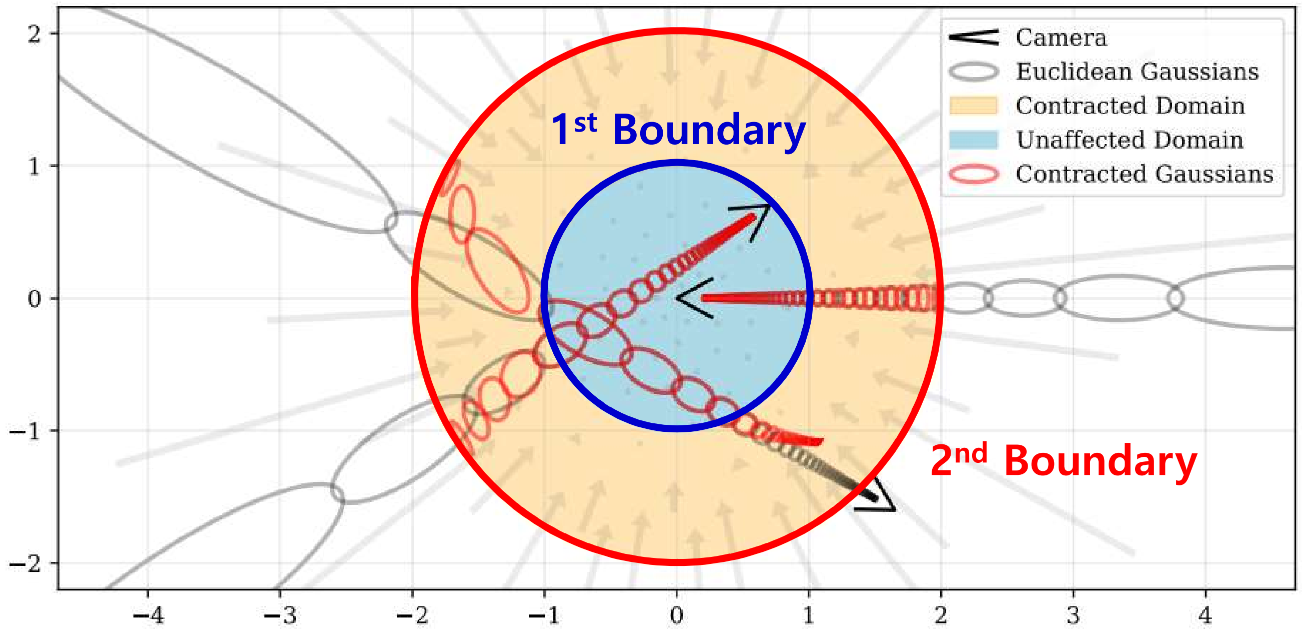
mipNeRF360: Rescaling for Unbounded scene

Parameterization

- Similar to NeRF++, set the **boundaries (but two)**. If samples exceed the 1st boundary then would be converged on 2nd boundary!



Contraction function f (linear transformation)

$$f(x) \approx f(\mu) + J_f(\mu)(x-\mu)$$

$$f(\mu,\Sigma) = \left(f(\mu),\ J_f(\mu)\Sigma\, J_f(\mu)^T \right)$$

$$f = \text{contract}(x) = \begin{cases} x & \|x\| \le 1 \\ \left(2 - \frac{1}{\|x\|}\right)\left(\frac{x}{\|x\|}\right) & \|x\| > 1 \end{cases}$$

1st Boundary ($\|x\| \le 1$)

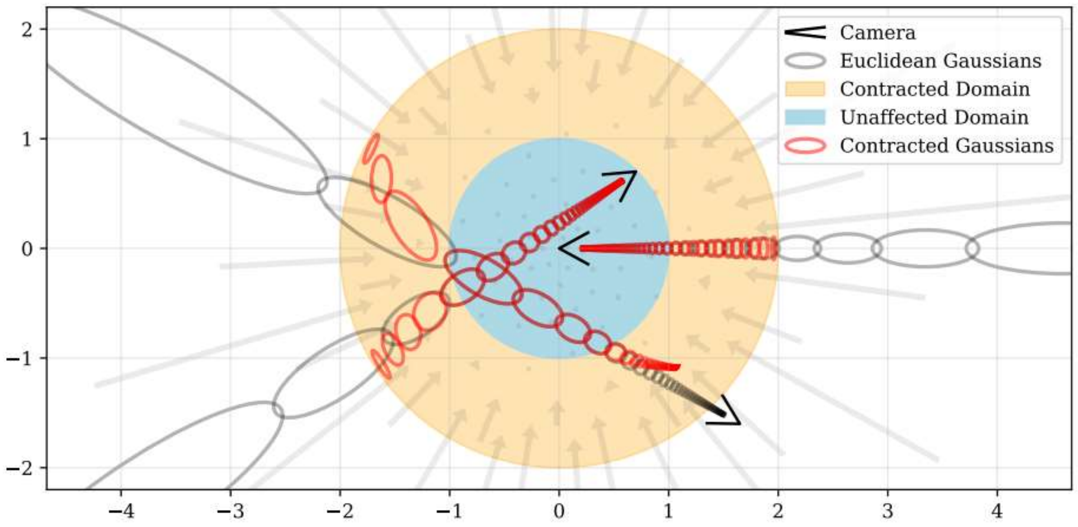
2nd Boundary (2)

PE of contraction: $\gamma\left(\text{contract}(\mu,\Sigma)\right)$

mipNeRF360: Sampling for Unbounded scene

Parameterization

- Similar to DONeRF, the farther away the wider the samples are
  - Uniform sampling on the **inverse of distance**
- Define **s-distance** to generalize sampling distance equation



$$s \in [0,1] \rightarrow [t_n, t_f]$$

$$s \triangleq \frac{g(t) - g(t_n)}{g(t_f) - g(t_n)}, \quad t \triangleq g^{-1}(s \cdot g(t_f) + (1-s) \cdot g(t_n))$$

t_n: near t, t_f: far t

$$g(x) = \frac{1}{x}$$

mipNeRF 360

$$g(x) = \log(x)$$

DONeRF spacing

mipNeRF360: Sampling for Unbounded scene

Parameterization

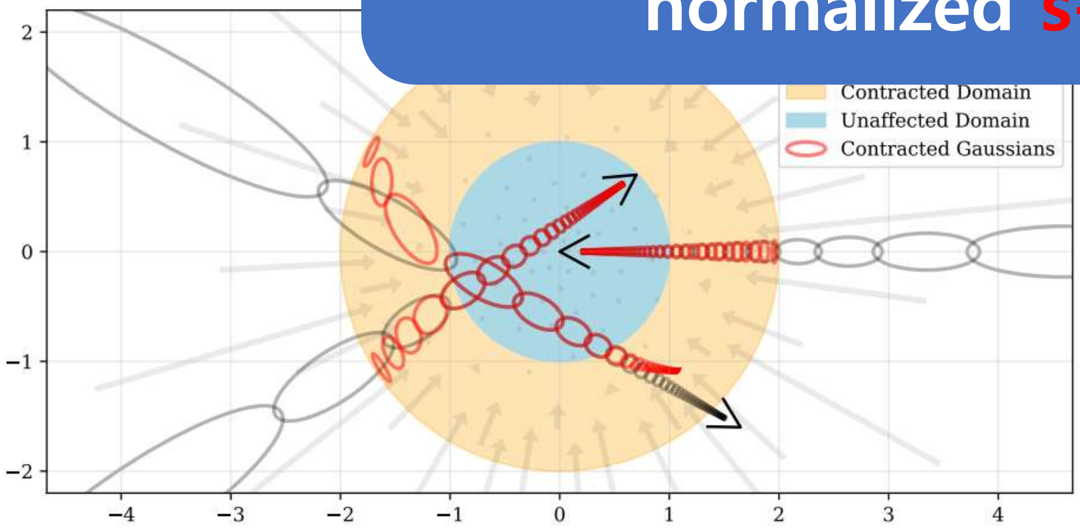
- Similar to DONeRF, the farther away the wider the samples are
  - Uniform
- Define **s-**

Contract coordinate effectively using 2 boundary warping and normalized s-distance

$[t_n, t_f]$

$$g(t_f) - g(t_n) \qquad (s \cdot g(t_f) + (1-s) \cdot g(t_n))$$

t_n: near t, t_f: far t



$g(x) = \frac{1}{x}$	$g(x) = \log(x)$
mipNeRF 360	**DONeRF spacing**

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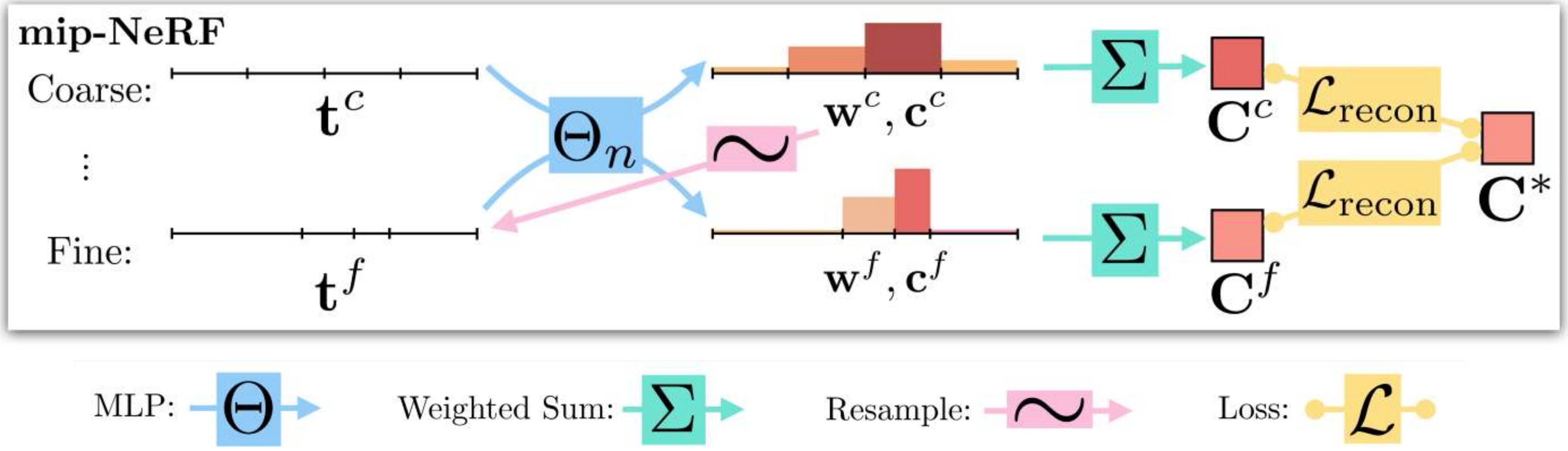
mipNeRF: Importance sampling

Efficiency

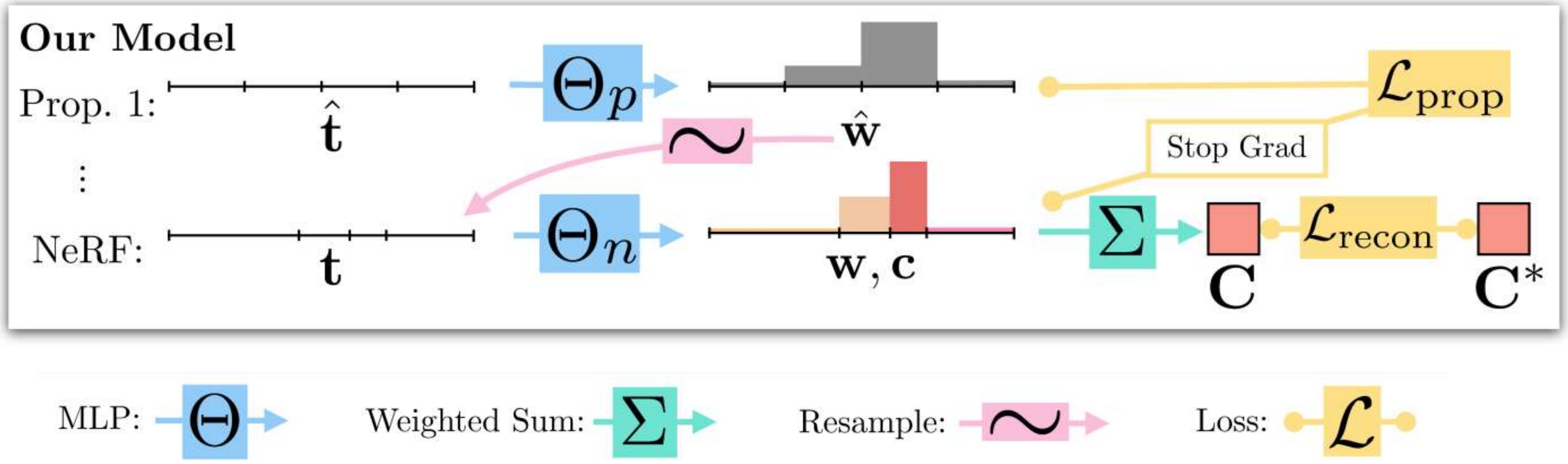
- First, **uniformly sample** the points and calculate **PDF** by its weight
- Second, perform **Importance sampling** based on the PDF of previous samples



mipNeRF360: Online distillation

Efficiency

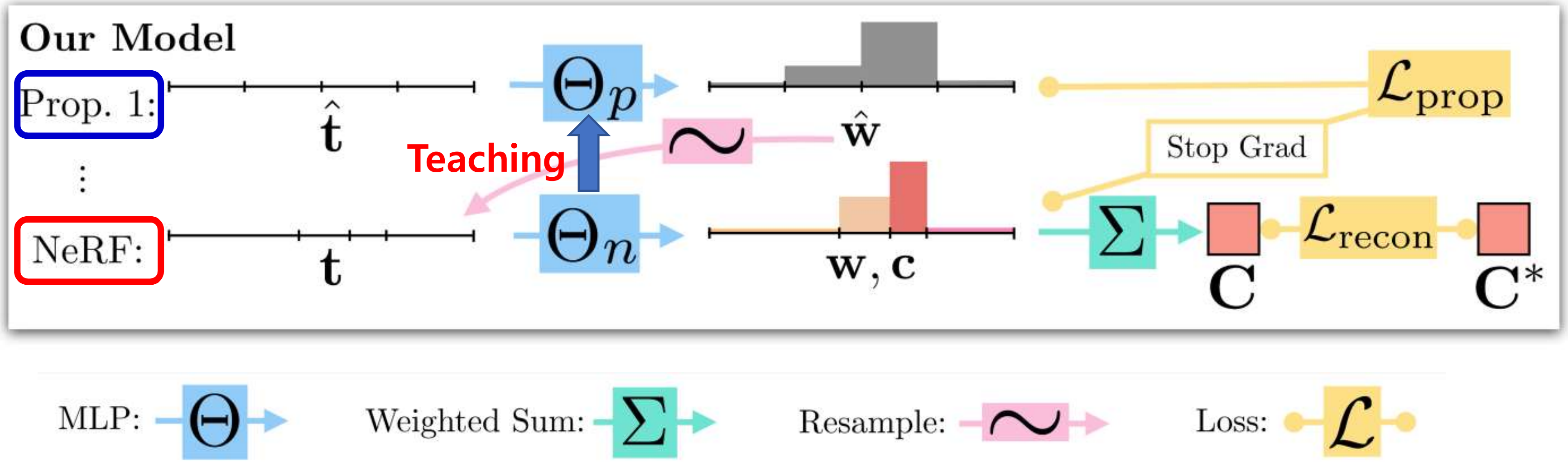
- **Distillation**: train a small model to imitate the huge model
  - Reduce the evaluation of huge model!



mipNeRF360: Online distillation

Efficiency

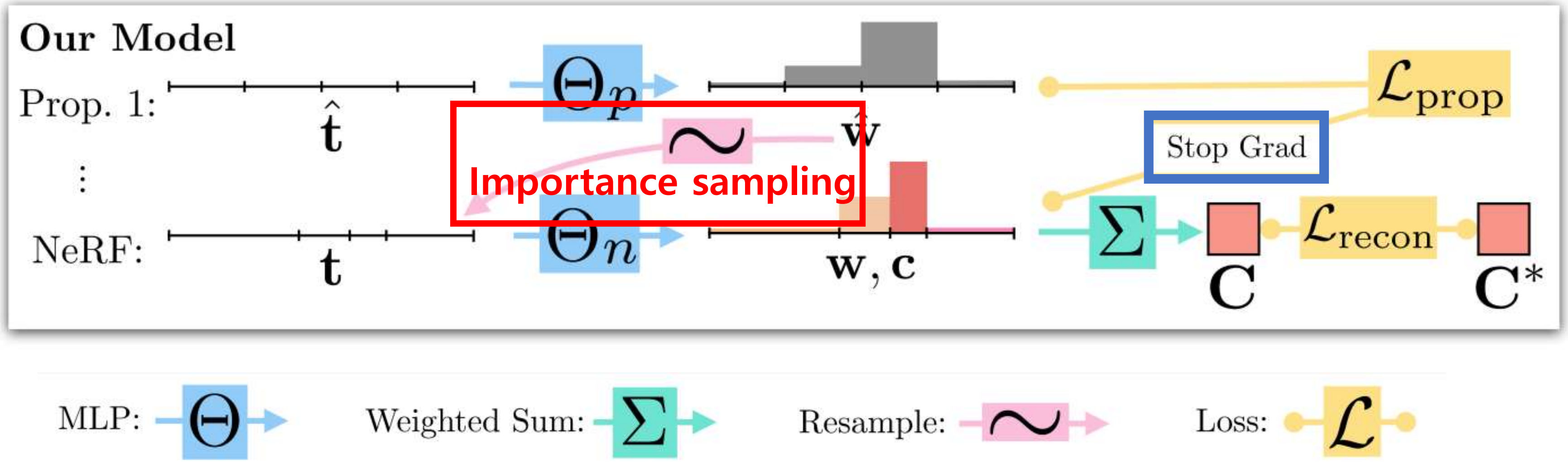
- Separate MLP as **proposal** MLP and **NeRF** MLP **(proposal ⋘ NeRF)**
- **Proposal** MLP does not predict the image directly
  - Only estimate the weights of samples



mipNeRF360: Online distillation

Efficiency

- Using **weights on proposal network** performs **importance sampling**
- Prevent the teaching MLP's update using "**Stop Gradient**" while updating student MLP

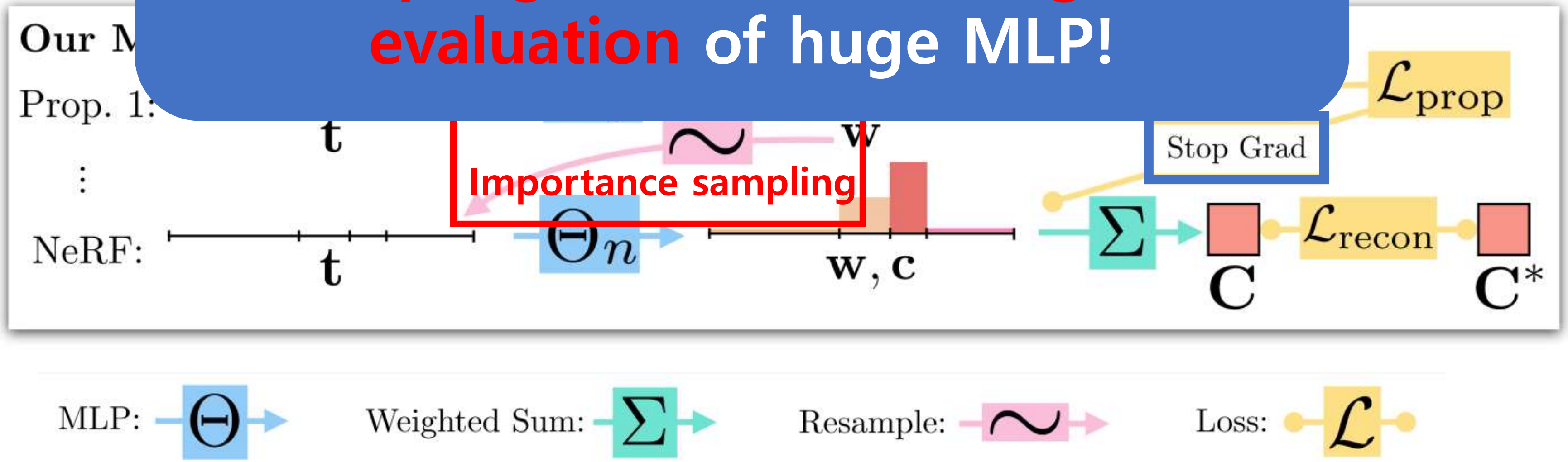


mipNeRF360: Online distillation

Efficiency

- Using **we** … **ampling**
- Prevent t … ile updating student N …

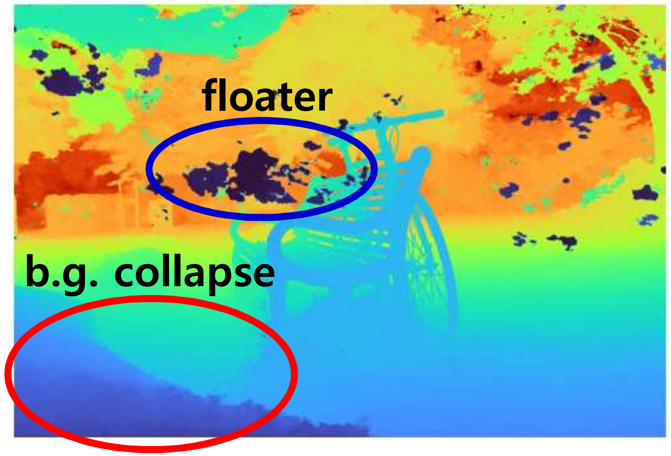
Online distillation enable mipNeRF360 to perform importance sampling while minimizing the evaluation of huge MLP!



mipNeRF: Characteristic Artifacts

Ambiguity

- **mipNeRF** does not have special function to discriminate **the surface or accurate geometry** -> 2 types of artifacts
  1. **Floater:** Floating discontinuity on a scene
  2. **Background Collapse:** predicting background as a set of semi-transparent cloud



Depth Result of **mipNeRF**

mipNeRF360: Regularization

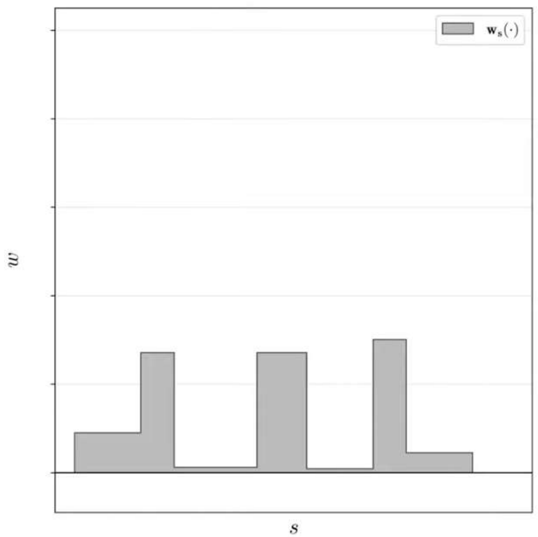
Ambiguity

- **Compress** the samples and its coefficients by adding **new loss**
- Make the weight distribution much sharper like delta function

  -> **Clarify the surface of contents**

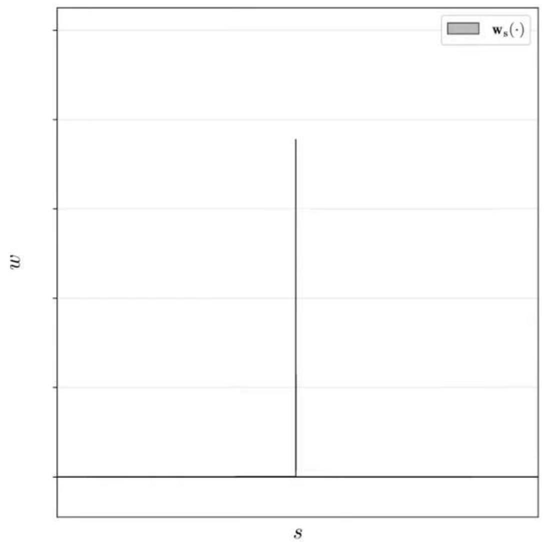
$$\mathcal{L}_{\text{dist}}(\mathbf{s}, \mathbf{w}) = \iint_{-\infty}^{\infty} \mathbf{w}_{\mathbf{s}}(u)\mathbf{w}_{\mathbf{s}}(v)|u - v| \, d_u \, d_v \,,$$

weight



Normalized s-distance

Distribution loss term



Normalized s-distance

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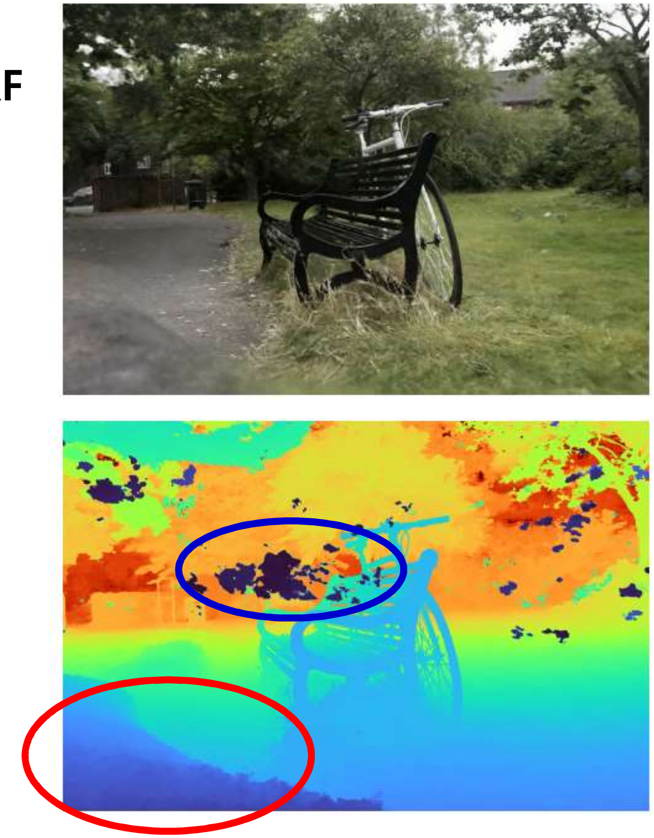
mipNeRF360: Regularization

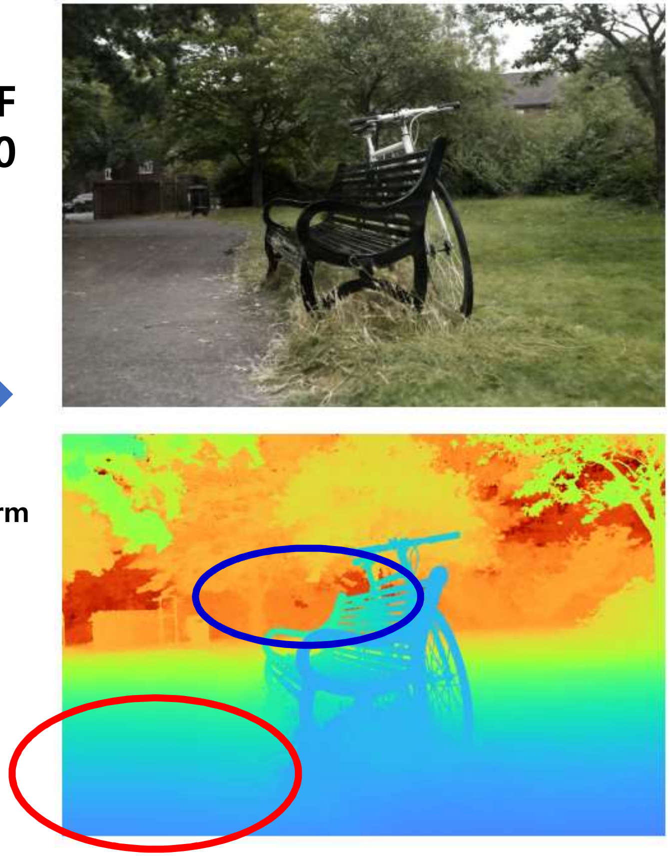
Ambiguity

- Successfully eliminate those artifacts!

mipNeRF



mipNeRF 360



Distribution loss term

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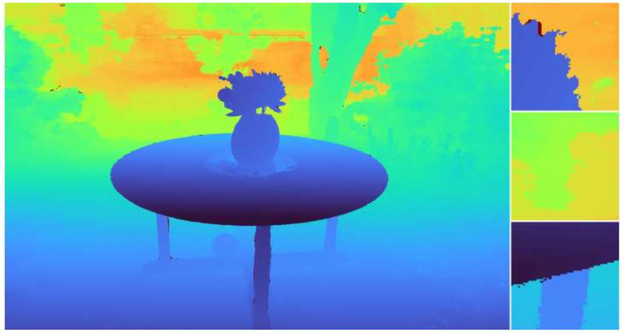
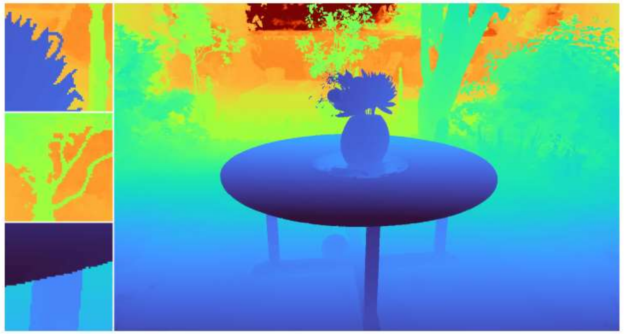
mipNeRF360: Result

- Shows outstanding result in 360 scene!



mipNeRF, SSIM: 0.526



mipNeRF360, SSIM: 0.804

PSNR

	Outdoor					Indoor			
	bicycle	*flowers*	*garden*	*stump*	*treehill*	*room*	*counter*	*kitchen*	*bonsai*
NeRF [13, 33]	21.76	19.40	23.11	21.73	21.28	28.56	25.67	26.31	26.81
NeRF w/ DONeRF [34] param.	21.67	19.48	23.29	23.38	21.70	28.28	25.74	25.42	27.32
mip-NeRF [3]	21.69	19.31	23.16	23.10	21.21	28.73	25.59	26.47	27.13
NeRF++ [51]	22.64	20.31	24.32	24.34	22.20	28.87	26.38	27.80	29.15
Deep Blending [17]	21.09	18.13	23.61	24.08	20.80	27.20	26.28	25.02	27.08
Point-Based Neural Rendering [26]	21.64	19.28	22.50	23.90	20.98	26.99	25.23	24.47	28.42
Stable View Synthesis [41]	22.79	20.15	25.99	24.39	21.72	28.93	26.40	28.49	29.07
mip-NeRF [3] w/bigger MLP	22.90	20.79	25.85	23.64	21.71	30.67	28.61	29.95	31.59
NeRF++ [51] w/bigger MLPs	23.75	21.11	25.91	25.48	22.77	30.13	27.79	29.85	30.68
Our Model	24.37	21.73	26.98	26.40	22.87	31.63	29.55	32.23	33.46
Our Model w/GLO	23.95	21.60	25.09	25.98	21.99	28.24	28.40	30.81	30.27

SSIM

	Outdoor					Indoor			
	bicycle	*flowers*	*garden*	*stump*	*treehill*	*room*	*counter*	*kitchen*	*bonsai*
NeRF [13, 33]	0.455	0.376	0.546	0.453	0.459	0.843	0.775	0.749	0.792
NeRF w/ DONeRF [34] param.	0.454	0.379	0.542	0.522	0.461	0.841	0.776	0.678	0.813
mip-NeRF [3]	0.454	0.373	0.543	0.517	0.466	0.851	0.779	0.745	0.818
NeRF++ [51]	0.526	0.453	0.635	0.594	0.530	0.852	0.802	0.816	0.876
Deep Blending [17]	0.466	0.320	0.675	0.634	0.523	0.868	0.856	0.768	0.883
Point-Based Neural Rendering [26]	0.608	0.487	0.735	0.651	0.579	0.887	0.868	0.876	0.919
Stable View Synthesis [41]	0.663	0.541	0.818	0.683	0.606	0.905	0.886	0.910	0.925
mip-NeRF [3] w/bigger MLP	0.612	0.514	0.777	0.643	0.577	0.903	0.877	0.902	0.928
NeRF++ [51] w/bigger MLPs	0.630	0.533	0.761	0.687	0.597	0.883	0.857	0.888	0.913
Our Model	0.685	0.583	0.813	0.744	0.632	0.913	0.894	0.920	0.941
Our Model w/GLO	0.687	0.582	0.800	0.745	0.619	0.907	0.890	0.916	0.932

mipNeRF360: Result

Summary of mipNeRF360

- Use **contraction** and **warping** to **normalize** the sample distance
- Use **online distillation** for **efficient** training
- Use **regularization** for solving **ambiguity** in unbounded scene

Total Loss :

$$\mathcal{L}_{\text{recon}}(\mathbf{C}(\mathbf{t}), \mathbf{C}^*) + \lambda \mathcal{L}_{\text{dist}}(\mathbf{s}, \mathbf{w}) + \sum_{k=0}^{1} \mathcal{L}_{\text{prop}}(\mathbf{s}, \mathbf{w}, \hat{\mathbf{s}}^k, \hat{\mathbf{w}}^k)$$

Ordinary image recon loss

Regularization for ambiguity

Loss for online distillation

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Thank you!

Quiz

1. IPE in mipNeRF can improve the performance without increasing the number of samples if they are sampled by importance sampling (True / False)

2. Why mipNeRF360 uses regularization in 360° scene? ( )
   ① For parameterization
   ② For importance sampling
   ③ For efficiency
   ④ For resolving the ambiguity

Reference

1. *Jonathan T. Barron, et al. Mip-NeRF: A Multiscale Representation for Anti-Aliasing Neural Radiance Fields*
2. *Jonathan T. Barron, et al. Mip-NeRF 360: Unbounded Anti-Aliased Neural Radiance Fields*
3. *https://jonbarron.info/mipnerf/*
4. *https://jonbarron.info/mipnerf360*
5. *https://www.youtube.com/watch?v=zBSH-k9GbV4*

CS580:
mipNeRF & mipNeRF360
Appendix

Integrated Positional Encoding (IPE)

Positional Encoding

$$\mathrm{P} = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 0 & & 2^{L-1} & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & \cdots & 0 & 2^{L-1} & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & & 0 & 0 & 2^{L-1} \end{bmatrix}^T, \quad \gamma(\mathrm{x}) = \begin{bmatrix} \sin(\mathrm{Px}) \\ \cos(\mathrm{Px}) \end{bmatrix}$$

Expectation

Positional encoding

$$E_{\mathrm{x} \sim N(\mu, \sigma^2)}[\gamma(\mathrm{x})]$$

Normal distribution

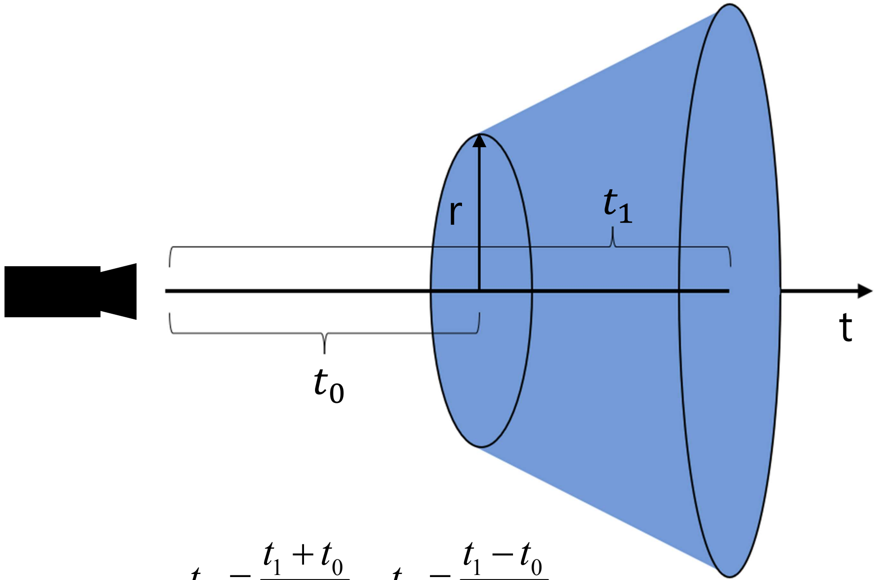
~

Conical frustum equation

$$F(x, o, d, \dot{r}, t_0, t_1) = \left\{ \left(t_0 < \frac{d^T(x-o)}{\|d\|_2^2} < t_1 \right) \wedge \left(\frac{d^T(x-o)}{\|d\|_2 \|x-o\|_2} > \frac{1}{\sqrt{1 + (\dot{r} / \|d\|_2)^2}} \right) \right\}$$

x are the member of conical frustum

Integrated Positional Encoding (IPE) (cont.)



$$t_\mu = \frac{t_1 + t_0}{2}, \quad t_\delta = \frac{t_1 - t_0}{2}$$

Mean, Covariance matrix of conical frustum

$$\mu_t = t_\mu + \frac{2t_\mu t_\delta^2}{3t_\mu^2 + t_\delta^2}, \quad \sigma_t^2 = \frac{t_\delta^2}{3} - \frac{4t_\delta^2(12t_\mu^2 - t_\delta^2)}{15(t_\mu^2 + t_\delta^2)^2},$$

$$\sigma_r^2 = \dot{r}^2\left(\frac{t_\mu^2}{4} + \frac{5t_\delta^2}{12} - \frac{4t_\delta^4}{15(3t_\mu^2 + t_\delta^2)}\right)$$

cone dir

radius dir

$$\mu = o + \mu_t d, \quad \Sigma = \sigma_t(dd^{\mathrm{T}}) + \sigma_r\left(I - \frac{dd^{\mathrm{T}}}{\|d\|_2^2}\right)$$

Proof of IPE

PDF

$$\varphi(r,t,\theta) = (rt\ cos\theta, rt\ sin\theta, t)$$

for $\theta \in [0,\ 2\pi),\ t \geq 0,\ |r| \leq \dot{r}$

$$dxdydz = |\det(D_\varphi)(r,t,\theta)|\, drdtd\theta$$

$$= \begin{vmatrix} t\cos\theta & t\sin\theta & 0 \\ r\cos\theta & r\sin\theta & 1 \\ -rt\sin\theta & rt\cos\theta & 0 \end{vmatrix} drdtd\theta$$

$$= (rt^2\cos^2\theta + rt^2\sin\theta)drdtd\theta$$

$$= rt^2 drdtd\theta$$

$$V = \int_0^{2\pi}\int_{t_0}^{t_1}\int_0^{\dot{r}} rt^2 drdtd\theta = \pi\dot{r}^2\frac{t_1^3 - t_0^3}{3}.$$

$$P = \frac{1}{V}.$$

Var of w.r.t. "t"

$$E[t^2] = \frac{1}{V}\int_0^{2\pi}\int_{t_0}^{t_1}\int_0^{\dot{r}} t^2 \cdot rt^2\, dr\, dt\, d\theta$$

$$= \frac{1}{V}\int_0^{2\pi}\int_{t_0}^{t_1}\int_0^{\dot{r}} rt^4\, dr\, dt\, d\theta$$

$$= \frac{1}{V}\cdot\pi\dot{r}^2\frac{t_1^5 - t_0^5}{5}$$

$$= \frac{3(t_1^5 - t_0^5)}{5(t_1^3 - t_0^3)}.$$

$$E[t] = \frac{1}{V}\int_0^{2\pi}\int_{t_0}^{t_1}\int_0^{\dot{r}} t \cdot rt^2\, dr\, dt\, d\theta$$

$$= \frac{1}{V}\int_0^{2\pi}\int_{t_0}^{t_1}\int_0^{\dot{r}} rt^3\, dr\, dt\, d\theta$$

$$= \frac{1}{V}\cdot\pi\dot{r}^2\frac{t_1^4 - t_0^4}{4}$$

$$= \frac{3(t_1^4 - t_0^4)}{4(t_1^3 - t_0^3)}.$$

$$Var(t) = \sigma_t^2 = E[t^2] - (E[t])^2$$

$$= \frac{3(t_1^5 - t_0^5)}{5(t_1^3 - t_0^3)} - \mu_t^2$$

Var of w.r.t. "x"(radius)

$$E[x^2] = \frac{1}{V}\int_0^{2\pi}\int_{t_0}^{t_1}\int_0^{\dot{r}} (rt\cos\theta)^2 \cdot rt^2\, dr\, dt\, d\theta$$

$$= \frac{1}{V}\int_{t_0}^{t_1}\int_0^{\dot{r}} r^3t^4 \int_0^{2\pi}\cos^2\theta\, d\theta\, dr\, dt$$

$$= \frac{1}{V}\cdot\frac{\dot{r}^4}{4}\cdot\frac{t_1^5 - t_0^5}{5}\cdot\pi$$

$$= \frac{\dot{r}^2}{4}\cdot\frac{3(t_1^5 - t_0^5)}{5(t_1^3 - t_0^3)}.$$

$$E[x] = \frac{1}{V}\int_0^{2\pi}\int_{t_0}^{t_1}\int_0^{r}(rt\cos\theta)\cdot rt\ drdtd\theta$$

$$= 0.$$

$$Var(x) = \sigma_r^2 = E[x^2] - (E[x])^2$$

$$= \dot{r}^2\left(\frac{3(t_1^5 - t_0^5)}{20(t_1^3 - t_0^3)}\right).$$

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Proof of IPE (conti.)

Positional encoding

$$P = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 0 & & 2^{L-1} & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & \cdots & 0 & 2^{L-1} & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & & 0 & 0 & 2^{L-1} \end{bmatrix}^T, \quad \gamma(x) = \begin{bmatrix} \sin(Px) \\ \cos(Px) \end{bmatrix}$$

$$x \rightarrow Px$$

$Cov[Ax,By] = A\, Cov[x,y]B^T$

$$\mu_\gamma = P\mu, \quad \Sigma_\gamma = P\Sigma P^T$$

Applying PE to x

Expectation of sin, cos following gaussian dist.

$$E_{x\sim N(\mu,\sigma^2)}[\sin(x)] = \sin(\mu)\exp(-(1/2)\sigma^2),$$
$$E_{x\sim N(\mu,\sigma^2)}[\cos(x)] = \cos(\mu)\exp(-(1/2)\sigma^2)$$

$$\gamma(\mu,\Sigma) = E_{x\sim N(\mu_\gamma,\Sigma_\gamma)}[\gamma(x)]$$
$$= \begin{bmatrix} \sin(\mu_\gamma)\circ\exp(-(1/2)diag(\Sigma_\gamma)), \\ \cos(\mu_\gamma)\circ\exp(-(1/2)diag(\Sigma_\gamma)), \end{bmatrix}$$

$$diag(\Sigma_\gamma) = \left[diag(\Sigma), 4diag(\Sigma), \cdots, 4^{L-1}diag(\Sigma)\right]^T$$
$$diag(\Sigma) = \sigma_t(d\circ d) + \sigma_r\left(I - \frac{d\circ d}{\|d\|_2^2}\right)$$

Proof of expectation of sin, cos following normal dist.

$$e^{ix} = \cos(x) + i\sin(x)$$ ········ **Euler formula**

$$E[e^{ix}] = E[\cos(x)] + iE[\sin(x)]$$

$$E_{x\sim \mathrm{N}(\mu,\sigma^2)}[e^{ix}] = \int_{-\infty}^{\infty} e^{ix} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \varphi_{x\sim N(\mu,\sigma^2)}(x)(t=1)$$ ········ **Characteristic function of normal dist.**

$$= e^{i\mu - \frac{1}{2}\sigma^2} \quad (\because e^{it\mu - \frac{1}{2}\sigma^2 t^2})$$

$$= \{\cos(\mu) + i\sin(\mu)\} e^{-\frac{1}{2}\sigma^2}$$

$$\therefore E[\cos(x)] = \cos(\mu) e^{-\frac{1}{2}\sigma^2}$$

$$E[\sin(x)] = \sin(\mu) e^{-\frac{1}{2}\sigma^2}$$

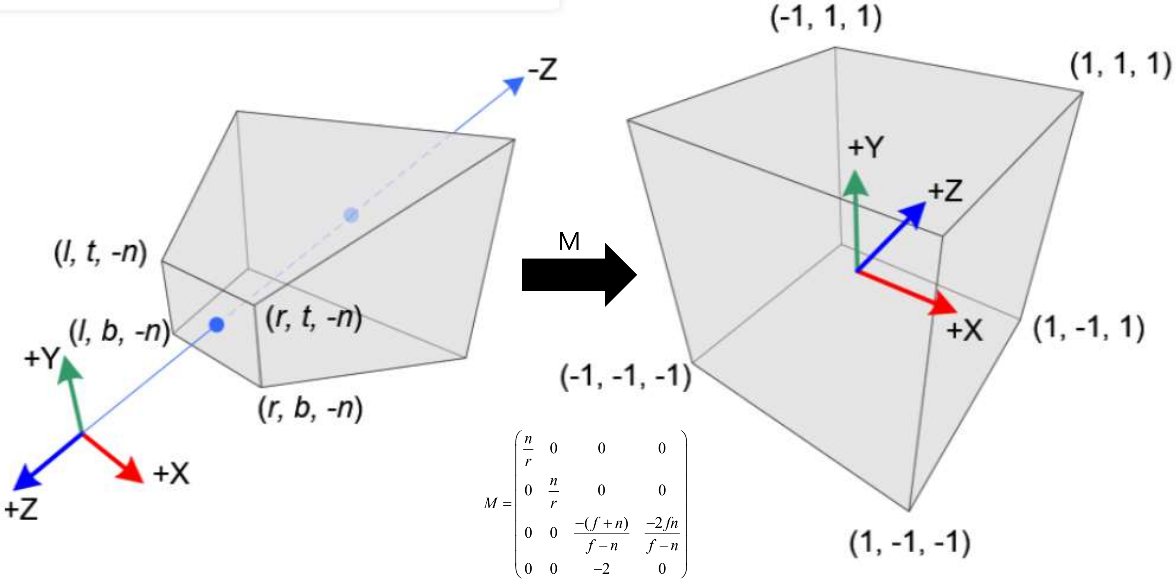
$$\therefore \quad E_{x\sim \mathrm{N}(\mu,\sigma^2)}[\sin(x)] = \sin(\mu)\exp(-(1/2)\sigma^2),$$

$$E_{x\sim \mathrm{N}(\mu,\sigma^2)}[\cos(x)] = \cos(\mu)\exp(-(1/2)\sigma^2)$$

NeRF in Unbounded scene

- ◆ 기존 NeRF dataset에서는 “facing forward” 라는 unbounded dataset이 존재
- ◆ 이를 효과적으로 렌더링하기 위해 ray의 원점과 방향을 계산한 후, Normalized device coordinate (NDC) 변환을 거친 다음 PE를 진행

NeRF applied to
“**facing forward scene**”



$$M = \begin{pmatrix} \frac{n}{r} & 0 & 0 & 0 \\ 0 & \frac{n}{r} & 0 & 0 \\ 0 & 0 & \frac{-(f+n)}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -2 & 0 \end{pmatrix}$$

Normalized Device Coordinate (**NDC**)

NDC in NeRF $(o+t\cdot d \;\rightarrow\; o'+t'\cdot d')$

r = right, t= top,
n = near, f = far

$$\begin{pmatrix} \frac{n}{r} & 0 & 0 & 0 \\ 0 & \frac{n}{t} & 0 & 0 \\ 0 & 0 & \frac{-(f+n)}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{n}{r}x \\ \frac{n}{t}y \\ \frac{-(f+n)}{f-n}z - \frac{-2fn}{f-n} \\ -z \end{pmatrix}$$

NDC projection matrix

$$\text{project} \rightarrow \begin{pmatrix} \frac{n}{r}\frac{x}{-z} \\ \frac{n}{t}\frac{y}{-z} \\ \frac{(f+n)}{f-n} - \frac{2fn}{f-n}\frac{1}{-z} \end{pmatrix}$$

$a_x = -\frac{n}{r},\ a_y = -\frac{n}{t},$
$a_z = -\frac{f+n}{f-n},\ b_z = -\frac{2fn}{f-n}$
$(x, y, z) = o + t\cdot d$

$$\begin{pmatrix} a_x \frac{o_x+td_x}{o_z+td_z} \\ a_y \frac{o_y+td_y}{o_z+td_z} \\ a_z + \frac{b_z}{o_z+td_z} \end{pmatrix} = \begin{pmatrix} o'_x + t'd'_x \\ o'_y + t'd'_y \\ o'_z + t'd'_z \end{pmatrix}$$

Suppose if t=0, then t'=0

$$\mathbf{o}' = \begin{pmatrix} o'_x \\ o'_y \\ o'_z \end{pmatrix} = \begin{pmatrix} a_x \frac{o_x}{o_z} \\ a_y \frac{o_y}{o_z} \\ a_z + \frac{b_z}{o_z} \end{pmatrix} = \pi(\mathbf{o})$$

$$\begin{pmatrix} t'd'_x \\ t'd'_y \\ t'd'_z \end{pmatrix} = \begin{pmatrix} a_x \frac{o_x+td_x}{o_z+td_z} - a_x \frac{o_x}{o_z} \\ a_y \frac{o_y+td_y}{o_z+td_z} - a_y \frac{o_y}{o_z} \\ a_z + \frac{b_z}{o_z+td_z} - a_z - \frac{b_z}{o_z} \end{pmatrix} = \begin{pmatrix} a_x \frac{o_z(o_x+td_x)-o_x(o_z+td_z)}{(o_z+td_z)o_z} \\ a_y \frac{o_z(o_y+td_y)-o_y(o_z+td_z)}{(o_z+td_z)o_z} \\ b_z \frac{o_z-(o_z+td_z)}{(o_z+td_z)o_z} \end{pmatrix}$$

$$= \begin{pmatrix} a_x \frac{td_z}{o_z+td_z}\left(\frac{d_x}{d_z} - \frac{o_x}{o_z}\right) \\ a_y \frac{td_z}{o_z+td_z}\left(\frac{d_y}{d_z} - \frac{o_y}{o_z}\right) \\ -b_z \frac{td_z}{o_z+td_z}\frac{1}{o_z} \end{pmatrix}$$

$$t' = \frac{td_z}{o_z+td_z} = 1 - \frac{o_z}{o_z+td_z}$$

$$\mathbf{d}' = \begin{pmatrix} a_x\left(\frac{d_x}{d_z} - \frac{o_x}{o_z}\right) \\ a_y\left(\frac{d_y}{d_z} - \frac{o_y}{o_z}\right) \\ -b_z \frac{1}{o_z} \end{pmatrix}.$$

https://blog.naver.com/PostView.naver?blogId=wnxodnr&logNo=10122064855&parentCategoryNo=26&categoryNo=&viewDate=&isShowPopularPosts=true&from=search NDC matrix proof

NDC in NeRF (cont.)

r = right, t= top, n = near, f = far

$$a_x = -\frac{n}{r},\ a_y = -\frac{n}{t},\ a_z = -\frac{f+n}{f-n},\ b_z = -\frac{2fn}{f-n}$$

Supposing pinhole camera

$f \to \infty$

$$a_x = -\frac{f_{cam}}{W/2},\ a_y = -\frac{f_{cam}}{H/2} \qquad a_z = 1,\ b_z = 2n$$

$$\mathbf{o}' = \begin{pmatrix} o'_x \\ o'_y \\ o'_z \end{pmatrix} = \begin{pmatrix} a_x \frac{o_x}{o_z} \\ a_y \frac{o_y}{o_z} \\ a_z + \frac{b_z}{o_z} \end{pmatrix} \qquad t' = \frac{td_z}{o_z + td_z} = 1 - \frac{o_z}{o_z + td_z}$$

$$\mathbf{d}' = \begin{pmatrix} a_x \left(\frac{d_x}{d_z} - \frac{o_x}{o_z} \right) \\ a_y \left(\frac{d_y}{d_z} - \frac{o_y}{o_z} \right) \\ -b_z \frac{1}{o_z} \end{pmatrix}.$$

$$\mathbf{o}' = \begin{pmatrix} -\frac{f_{cam}}{W/2} \frac{o_x}{o_z} \\ -\frac{f_{cam}}{H/2} \frac{o_y}{o_z} \\ 1 + \frac{2n}{o_z} \end{pmatrix} \qquad t' = \frac{td_z}{o_z + td_z} = 1 - \frac{o_z}{o_z + td_z}$$

$$\mathbf{d}' = \begin{pmatrix} -\frac{f_{cam}}{W/2} \left(\frac{d_x}{d_z} - \frac{o_x}{o_z} \right) \\ -\frac{f_{cam}}{H/2} \left(\frac{d_y}{d_z} - \frac{o_y}{o_z} \right) \\ -2n \frac{1}{o_z} \end{pmatrix}$$

NDC Projection $o + t \cdot d \ \to \ o' + t' \cdot d'$ ($t \sim (0, \infty) \to t' \sim (0,1)$)

https://blog.naver.com/PostView.naver?blogId=wnxodnr&logNo=10122064855&parentCategoryNo=26&categoryNo=&viewDate=&isShowPopularPosts=true&from=search NDC matrix proof